Identical Particles

Quantum Mechanics II

Ding-Yu Shao
dyshao@fudan.edu.cn

Department of physics, Fudan University

Contents:

Exchange Degeneracy

Identical Particles

- Two particles are identical if all their intrinsic properties (mass, spin, charge, magnetic moment, etc.) are the same and therefore no experiment can distinguish them.
- Two identical particles can have different momentum, energy, angular momentum.
  - For example all electrons are identical, all protons, all neutrons, all hydrogen atoms are identical (the possible excitation states viewed as energy, momentum, etc.)

Exchange Degeneracy

- In quantum mechanics, the states of **identical particles** face a unique complication: **exchange degeneracy**.
- This degeneracy arises because identical particles cannot be distinguished by labels, leading to equivalent choices in state descriptions.
- For **distinguishable particles**, we can clearly write the state as:

$$|v\rangle_1 \otimes |w\rangle_2 \in V \otimes W.$$

- However, for **identical particles**, labeling is ambiguous. For example (focus only on the spin):

$$|+\rangle_1 \otimes |-\rangle_2 \quad \text{and} \quad |-\rangle_1 \otimes |+\rangle_2$$

  both describe the same physical system.
- Two different states in $V \otimes V$; in fact, they are orthogonal states. The labels are useful for purposes of description: for the first state, we say that the first particle is up and the second down; for the second state, the first particle is down, and the second particle is up.

Superposition and Continuous Degeneracy

- If the two states above are equivalent, any **superposition** of them is also an equivalent description:
$$|\psi\rangle = \alpha\,|+\rangle_1 \otimes |-\rangle_2 + \beta\,|-\rangle_1 \otimes |+\rangle_2\,, \quad |\alpha|^2 + |\beta|^2 = 1.$$
- The states are equivalent for all possible values of α and β. This results in **continuous degeneracy**
- This degeneracy complicates the specification of a state for identical particles.

Ambiguity Example and Exchange Degeneracy Problem

- To illustrate the ambiguity, consider calculating the probability p_0 of finding both particles in a specific state $|\psi_0\rangle$:
$$|\psi_0\rangle = |x;+\rangle_1 \otimes |x;+\rangle_2$$
- Alternatively, $|\psi_0\rangle$ can be written in terms of the usual z-based states
$$\begin{aligned}|\psi_0\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle_{(1)} + |-\rangle_{(1)}\right) \otimes \frac{1}{\sqrt{2}}\left(|+\rangle_{(2)} + |-\rangle_{(2)}\right) \\ &= \frac{1}{2}\Big(|+\rangle_{(1)} \otimes |+\rangle_{(2)} + |+\rangle_{(1)} \otimes |-\rangle_{(2)} + |-\rangle_{(1)} \otimes |+\rangle_{(2)} + |-\rangle_{(1)} \otimes |-\rangle_{(2)}\Big).\end{aligned}$$
- The probability p_0 of finding the particles in state $|\psi_0\rangle$ is given by:
$$p_0 = |\langle \psi_0 \mid \psi\rangle|^2 = \left|\frac{1}{2}(\alpha + \beta)\right|^2.$$

Ambiguity in the Probability Calculation

- This probability calculation reveals ambiguity depending on the choice of α and β:

$$\text{If } \alpha = \beta = \frac{1}{\sqrt{2}}, \quad \text{then } p_0 = \frac{1}{2}$$

- However, with a different choice:

$$\text{If } \alpha = -\beta = \frac{1}{\sqrt{2}}, \quad \text{then } p_0 = 0$$

- The chosen values of α and β matter when they should not.
- This ambiguity illustrates the problem of **exchange degeneracy**.
- The symmetrization postulate gives a satisfactory resolution of the exchange degeneracy conundrum, stating that physical states are either totally symmetric or totally antisymmetric under permutations of the particles. Particles described by symmetric states are called **bosons**, and particles described by antisymmetric states are called **fermions**.

Permutation Operators

Permutation Operators

- We will consider permutation operators and the permutation groups they form. These are relevant when we have identical particles that can be exchanged without physical consequence.
- These operators act on the tensor-product space:

$$V^{\otimes N} \equiv \underbrace{V \otimes \cdots \otimes V}_{N}, , \tag{1}$$

  which is the candidate state space for the description of N identical particles.
- In this section, we do not yet implement the constraints due to identical particles. The only constraint we are imposing is that each of the particles in the N-particle collection lives in a state space V; thus $V^{\otimes N}$ is the state space relevant to the collection.
- For the moment, it is best to think of the particles as distinguishable.

Two-particle systems: Overview

- **Two-particle systems:** Assume we have two particles, particle one and particle two, each described by the same state space V spanned by orthonormal basis states $|u_i\rangle$, with $i = 1, 2, \ldots$.
- Consider the two-particle state where particle one is in state $|u_i\rangle$ and particle two is in state $|u_j\rangle$, with $u_i \neq u_j$:
$$|u_i\rangle_{(1)} \otimes |u_j\rangle_{(2)} \in V \otimes V .$$
- Note: $|u_j\rangle_{(1)} \otimes |u_i\rangle_{(2)}$ is a different state in $V \otimes V$, where particle one is in $|u_j\rangle$ and particle two is in $|u_i\rangle$.

Permutation Operator

- We define a linear operator $\hat{P}_{21}$ acting on $V \otimes V$ that produces this exchange by permuting the i and j labels:

$$\hat{P}_{21}|u_i\rangle_{(1)} \otimes |u_j\rangle_{(2)} = |u_j\rangle_{(1)} \otimes |u_i\rangle_{(2)}.$$

- This describes the action of $\hat{P}_{21} \in \mathcal{L}(V \otimes V)$ on all basis vectors of $V \otimes V$, fully defining the operator.
- $\hat{P}_{21}$ is called a **transposition** operator, as it transposes the states of particle one and particle two.

Properties of the Transposition Operator

- The iterated action of $\hat{P}_{21}$ gives the identity operator:

$$\hat{P}_{21}^2 = 1.$$

- This means that $\hat{P}_{21}$ is its own inverse.
- We now claim that the operator $\hat{P}_{21}$ is Hermitian:

$$\hat{P}_{21}^\dagger = \hat{P}_{21}.$$

Proof of Hermiticity

- **Proof:** An operator $\hat{M}$ is Hermitian if $\langle \hat{M}\alpha, \beta\rangle = \langle \alpha, \hat{M}\beta\rangle$ for all states α, β.
- Writing the states as $u_i \otimes u_j = |u_i\rangle_{(1)} \otimes |u_j\rangle_{(2)}$, we have the inner product:

$$\langle u_k \otimes u_l | u_i \otimes u_j\rangle = \delta_{ik}\delta_{jl}.$$

- We check that

$$\langle u_k \otimes u_l, \hat{P}_{21} u_i \otimes u_j\rangle = \langle u_k \otimes u_l, u_j \otimes u_i\rangle = \delta_{il}\delta_{jk},$$

- is equal to

$$\langle \hat{P}_{21} u_k \otimes u_l, u_i \otimes u_j\rangle = \langle u_l \otimes u_k, u_i \otimes u_j\rangle = \delta_{il}\delta_{jk}.$$

- This confirms Hermiticity.

Unitarity of the Transposition Operator

- Since the transposition operator is Hermitian and squares to itself, it is also unitary:

$$\hat{P}_{21}\hat{P}_{21}^{\dagger} = 1.$$

- This suggests that the transposition operator could be used to define a symmetry.

Eigenvalues and Eigenstates of $\hat{P}_{21}$

- Given a Hermitian operator, such as $\hat{P}_{21}$, we consider its eigenvalues and eigenvectors.
- Since $\hat{P}_{21}^2 = 1$, the eigenvalues λ of $\hat{P}_{21}$ must satisfy $\lambda^2 = 1$. Thus, $\lambda = \pm 1$ are the only possibilities.
- The corresponding eigenstates in $V \otimes V$ are classified as:
  - If $\hat{P}_{21}|\psi\rangle = |\psi\rangle$, then $|\psi\rangle$ is a **symmetric state**.
  - If $\hat{P}_{21}|\psi\rangle = -|\psi\rangle$, then $|\psi\rangle$ is an **antisymmetric state**.
- Symmetric and antisymmetric states are, respectively, invariant or change sign under transpositions.
- The set of symmetric states forms a subspace $\mathrm{Sym}(V \otimes V)$, and the antisymmetric states form $\mathrm{Anti}(V \otimes V)$.

Operators $\hat{S}$ and $\hat{A}$

- These subspaces are constructed using two Hermitian operators $\hat{S}$ and $\hat{A}$, defined as:

$$\hat{S} \equiv \frac{1}{2}(1 + \hat{P}_{21}), \quad \hat{A} \equiv \frac{1}{2}(1 - \hat{P}_{21}).$$

- The operators $\hat{S}$ and $\hat{A}$ satisfy:

$$\hat{P}_{21}\hat{S} = \hat{S} \quad \text{and} \quad \hat{P}_{21}\hat{A} = -\hat{A}.$$

- Verification of these identities:

$$\hat{P}_{21}\hat{S} = \frac{1}{2}(\hat{P}_{21} + \hat{P}_{21}^2) = \frac{1}{2}(\hat{P}_{21} + 1) = \hat{S},$$

$$\hat{P}_{21}\hat{A} = \frac{1}{2}(\hat{P}_{21} - \hat{P}_{21}^2) = \frac{1}{2}(\hat{P}_{21} - 1) = -\hat{A}.$$

- This confirms that $\hat{S}$ projects onto symmetric states and $\hat{A}$ onto antisymmetric states.

Symmetric and Antisymmetric States

- Given a generic state $|\psi\rangle \in V \otimes V$, we define:

$$|\psi_S\rangle = \hat{S}|\psi\rangle \in \mathrm{Sym}(V \otimes V), \quad |\psi_A\rangle = \hat{A}|\psi\rangle \in \mathrm{Anti}(V \otimes V).$$

- This gives us a symmetric state $|\psi_S\rangle$ and an antisymmetric state $|\psi_A\rangle$.
- We verify that:

$$\hat{P}_{21}|\psi_S\rangle = \hat{P}_{21}\hat{S}|\psi\rangle = \hat{S}|\psi\rangle = |\psi_S\rangle,$$

$$\hat{P}_{21}|\psi_A\rangle = \hat{P}_{21}\hat{A}|\psi\rangle = -\hat{A}|\psi\rangle = -|\psi_A\rangle.$$

- These relations confirm that $\hat{S}$ acts as a projector onto the symmetric subspace $\mathrm{Sym}(V \otimes V)$, and $\hat{A}$ acts as a projector onto the antisymmetric subspace $\mathrm{Anti}(V \otimes V)$.

Projector Properties of $\hat{S}$ and $\hat{A}$

- The Hermitian operator $\hat{S}$ projects onto $\mathrm{Sym}(V \otimes V)$ and $\hat{A}$ projects onto $\mathrm{Anti}(V \otimes V)$.
- These projectors satisfy:
$$\hat{S}\hat{S} = \hat{S}, \quad \hat{A}\hat{A} = \hat{A}.$$
- Additionally, as complementary projectors, they obey:
$$\hat{S} + \hat{A} = 1, \quad \hat{S}\hat{A} = \hat{A}\hat{S} = 0.$$
- The full space $V \otimes V$ can be decomposed into the orthogonal subspaces of symmetric and antisymmetric states.
$$V \otimes V = \text{ range } \hat{S} \oplus \text{ range } \hat{A}, \quad \text{ and } \quad \text{ range } \hat{S} \perp \text{ range } \hat{A}.$$

N-Particle Systems and Permutation Operators

- For $N = 2$, we have a simple permutation group with only the identity operator and a single permutation operator $\hat{P}_{21}$.
- The symmetric group S_2 consists of:
  - Identity operator 1.
  - Transposition $\hat{P}_{21}$, which exchanges the states of two particles.
- For $N > 2$, the number of permutation operators increases, forming the symmetric group S_N, where permutations can involve rearranging more particles.

Permutation Notation for N-Particle Systems

- Define permutation operators for N particles using the notation $\hat{P}_{i_1 i_2 \ldots i_N}$, where:
  - $\{i_1, i_2, \ldots, i_N\}$ represents a reordering of particles.
  - $\hat{P}_{ijk}$ moves the state of the i-th particle to the first position, j-th particle to the second, and k-th particle to the third.
- For example:

$$\hat{P}_{231} \left|u_r\right\rangle_{(1)} \otimes \left|u_s\right\rangle_{(2)} \otimes \left|u_t\right\rangle_{(3)} = \left|u_s\right\rangle_{(1)} \otimes \left|u_t\right\rangle_{(2)} \otimes \left|u_r\right\rangle_{(3)} .$$

- The inverse of $\hat{P}_{231}$ is $\hat{P}_{312}$ such that:

$$\hat{P}_{231} \hat{P}_{312} = 1.$$

Permutation Operators

- A permutation of N objects is defined by the function α that maps the ordered integers $1, \ldots, N$ into an arbitrary ordering:

$$\alpha : [1, \ldots, N] \to [\alpha(1), \ldots, \alpha(N)].$$

- The permutation operator $\hat{P}_\alpha$ associated with α is written as:

$$\hat{P}_\alpha \equiv \hat{P}_{\alpha(1),\alpha(2),\ldots,\alpha(N)}.$$

- Extending the rule for the three-particle case, this operator acts as follows:

$$\hat{P}_\alpha |u_1\rangle_{(1)} \otimes \cdots \otimes |u_N\rangle_{(N)} = |u_{\alpha(1)}\rangle_{(1)} \otimes \cdots \otimes |u_{\alpha(N)}\rangle_{(N)}.$$

- Example:

$$\hat{P}_{3142} |u_1\rangle_{(1)} \otimes |u_2\rangle_{(2)} \otimes |u_3\rangle_{(3)} \otimes |u_4\rangle_{(4)} = |u_3\rangle_{(1)} \otimes |u_1\rangle_{(2)} \otimes |u_4\rangle_{(3)} \otimes |u_2\rangle_{(4)}.$$

- Dropping subscripts on states and assuming they are ordered from 1 to N, we write:

$$\hat{P}_{3142} |a\rangle \otimes |b\rangle \otimes |c\rangle \otimes |d\rangle = |c\rangle \otimes |a\rangle \otimes |d\rangle \otimes |b\rangle.$$

- For simplicity, omitting the tensor product symbol $\otimes$, we have:

$$\hat{P}_{3142} |abcd\rangle = |cadb\rangle.$$

Symmetric Group S_3

- The $N = 3$ permutation operators form the symmetric group S_3 with $3! = 6$ elements:

$$\hat{P}_{123} = 1, \quad \hat{P}_{312}, \quad \hat{P}_{231}, \quad \underbrace{\hat{P}_{132}, \quad \hat{P}_{213}, \quad \hat{P}_{321}}_{\text{transpositions}}.$$

- A transposition is a permutation in which only two labels are exchanged, leaving the rest in canonical order. For example:
  - $\hat{P}_{132}$: Swaps the states of the second and third particles, leaving the first particle unchanged.
- Transpositions can be denoted by the labels of the two exchanged particles in ascending order:

$$(23) = \hat{P}_{132}, \quad (12) = \hat{P}_{213}, \quad (13) = \hat{P}_{321}.$$

Symmetric Group S_3 (Part 2)

- The multiplication table for the group S_3 is provided below.

Table: A · B matrix for S_3.

A \ B	1	$\hat{P}_{312}$	$\hat{P}_{231}$	(23)	(12)	(13)
1	1	$\hat{P}_{312}$	$\hat{P}_{231}$	(23)	(12)	(13)
$\hat{P}_{312}$	$\hat{P}_{312}$	$\hat{P}_{231}$	1	(12)	(13)	(23)
$\hat{P}_{231}$	$\hat{P}_{231}$	1	$\hat{P}_{312}$	(13)	(23)	(12)
(23)	(23)	(13)	(12)	1	$\hat{P}_{231}$	$\hat{P}_{312}$
(12)	(12)	(23)	(13)	$\hat{P}_{312}$	1	$\hat{P}_{231}$
(13)	(13)	(12)	(23)	$\hat{P}_{231}$	$\hat{P}_{312}$	1

Properties of Permutations

- In general, any permutation can be expressed as a product of transpositions.
- Any set of integers can be rearranged into any arbitrary position through successive transpositions.
- The decomposition of a permutation into transpositions is not unique, but the parity of the permutation (even or odd) is unique modulo 2:
  - A permutation is **even** if it results from an even number of transpositions.
  - A permutation is **odd** if it results from an odd number of transpositions.
- The identity element is an even permutation.
- Even permutations have even parity, while odd permutations have odd parity.

Hermitian and Unitary Properties of Transpositions

- All transpositions are Hermitian and unitary. This proof extends to general permutations:
  - Since the product of unitary operators is unitary, any permutation is a unitary operator.
  - Arbitrary products of transpositions are not Hermitian unless they commute.
- Unitarity implies that the Hermitian conjugate of a permutation is its inverse, preserving parity:

$$\hat{P}_\alpha = \hat{P}_{t_1} \dots \hat{P}_{t_k} \to \hat{P}_\alpha^\dagger = \hat{P}_{t_k}^\dagger \dots \hat{P}_{t_1}^\dagger = \hat{P}_{t_k} \dots \hat{P}_{t_1}$$

  and thus, as expected,

$$\hat{P}_\alpha \hat{P}_\alpha^\dagger = 1.$$

- Claim: For any S_N, the number of even permutations equals the number of odd permutations.
  - Proof outline: Map m_{12} transforms even to odd permutations and vice versa. This map is one-to-one, confirming equal counts.

Symmetrizer and Antisymmetrizer

Complete Symmetrizer and Antisymmetrizer

- Permutation operators do not all commute, so we cannot generally find a complete basis of states that are eigenstates of all permutation operators.
  - However, we can find some states that are simultaneous eigenstates of all permutation operators.
  - Lack of commuting operators means no complete basis but allows some simultaneous eigenstates.
- Consider N particles, each with state space V, forming a collection in $V^{\otimes N}$.
  - Define **symmetric states** $|\psi_S\rangle$ that remain invariant under all permutations $\hat{P}_\alpha$:

$$\hat{P}_\alpha|\psi_S\rangle = |\psi_S\rangle, \quad \forall\alpha.$$

  - Symmetric states are eigenstates with eigenvalue $+1$ and form the subspace $\mathrm{Sym}^N V$.
  - Define **antisymmetric states** $|\psi_A\rangle$, where odd permutations change the sign:

$$\hat{P}_\alpha|\psi_A\rangle = \epsilon_\alpha|\psi_A\rangle, \quad \epsilon_\alpha = \begin{cases} +1, & \text{if } \hat{P}_\alpha \text{ is even} \\ -1, & \text{if } \hat{P}_\alpha \text{ is odd} \end{cases}$$

  - Antisymmetric states form the subspace $\mathrm{Anti}^N V \subset V^{\otimes N}$.

Projectors for Symmetric and Antisymmetric Subspaces

Our next goal is the construction of projectors $\hat{S}$ and $\hat{A}$ from $V^{\otimes N}$ into $\mathrm{Sym}^N V$ and $\mathrm{Anti}^N V$, respectively. As we will confirm below, the projectors are defined as follows:

$$\hat{S} \equiv \frac{1}{N!} \sum_{\alpha} \hat{P}_\alpha \quad \text{and} \quad \hat{A} \equiv \frac{1}{N!} \sum_{\alpha} \epsilon_\alpha \hat{P}_\alpha .$$

where the sums are over all $N!$ permutations. $\hat{S}$ is called the symmetrizer, and $\hat{A}$ is called the antisymmetrizer. The first thing to note is that both $\hat{S}$ and $\hat{A}$ are Hermitian operators:

$$\hat{S} = \hat{S}^\dagger \quad \text{and} \quad \hat{A} = \hat{A}^\dagger .$$

Hermitian Conjugation and Permutations

Hermitian conjugation is a one-to-one invertible map from the set of all permutations to itself. Due to unitarity, Hermitian conjugation maps each permutation to its inverse. Since the sum in $\hat{S}$ is simply reordered, making it clear that $\hat{S}$ is not changed.
Given an α permutation, we define the $\alpha^\dagger$ permutation via

$$P_\alpha^\dagger \equiv P_{\alpha^\dagger}. \tag{2}$$

Here $\alpha^\dagger : [\alpha^\dagger(1), \ldots, \alpha^\dagger(N)]$ is the list that makes the above equation hold. Since the set of all $\alpha^\dagger$'s is equal to the set of all α's, it follows that for any function $f(\alpha)$ of α we have

$$\sum_\alpha f(\alpha^\dagger) = \sum_\alpha f(\alpha). \tag{3}$$

Indeed, both sides of the equation simply compute the sum of the evaluation of f over all permutations.

Hermitian Conjugation of $\hat{S}$ and $\hat{A}$

Now consider the Hermitian conjugation of $\hat{S}$. We have

$$\hat{S}^\dagger = \frac{1}{N!}\sum_\alpha \hat{P}_\alpha^\dagger = \frac{1}{N!}\sum_\alpha \hat{P}_{\alpha^\dagger}. \tag{4}$$

We now use the identity (3) to find that

$$\hat{S}^\dagger = \hat{S}. \tag{5}$$

The antisymmetrizer $\hat{A}$ is also unchanged because Hermitian conjugation does not change the parity of a permutation—namely, $\epsilon_{\alpha^\dagger} = \epsilon_\alpha$. As a result,

$$\hat{A}^\dagger = \frac{1}{N!}\sum_\alpha \epsilon_\alpha \hat{P}_{\alpha^\dagger} = \frac{1}{N!}\sum_\alpha \epsilon_{\alpha^\dagger} \hat{P}_{\alpha^\dagger} = \frac{1}{N!}\sum_\alpha \epsilon_\alpha \hat{P}_\alpha = \hat{A}, \tag{6}$$

It is also important to see what happens when $\hat{S}$ or $\hat{A}$ is multiplied by a permutation operator. We claim that

$$\hat{P}_{\alpha_0}\hat{S} = \hat{S}\hat{P}_{\alpha_0} = \hat{S}, \quad \hat{P}_{\alpha_0}\hat{A} = \hat{A}\hat{P}_{\alpha_0} = \epsilon_{\alpha_0}\hat{A}. \tag{7}$$

Orthogonal Projectors: $\hat{S}$ and $\hat{A}$

Both $\hat{S}$ and $\hat{A}$ are orthogonal projectors:

$$\hat{S}^2 = \hat{S}, \quad \hat{A}^2 = \hat{A}, \quad \hat{S}\hat{A} = \hat{A}\hat{S} = 0. \tag{8}$$

Proof:

$$\hat{S}^2 = \frac{1}{N!}\sum_{\alpha} \hat{P}_\alpha \hat{S} = \frac{1}{N!}\sum_{\alpha} \hat{S} = \frac{1}{N!} N! \hat{S} = \hat{S}, \tag{9}$$

$$\hat{A}^2 = \frac{1}{N!}\sum_{\alpha} \epsilon_\alpha \hat{P}_\alpha \hat{A} = \frac{1}{N!}\sum_{\alpha} \epsilon_\alpha \epsilon_\alpha \hat{A} = \frac{1}{N!}\sum_{\alpha} \hat{A} = \frac{1}{N!} N! \hat{A} = \hat{A}, \tag{10}$$

$$\hat{A}\hat{S} = \frac{1}{N!}\sum_{\alpha} \epsilon_\alpha \hat{P}_\alpha \hat{S} = \frac{1}{N!}\sum_{\alpha} \epsilon_\alpha \hat{S} = \frac{\hat{S}}{N!}\sum_{\alpha} \epsilon_\alpha = 0. \tag{11}$$

Here, $\sum_\alpha \epsilon_\alpha = 0$ follows because there are equal numbers of even and odd permutations.

Projection to Symmetric and Antisymmetric States

We now confirm that $\hat{S}$ and $\hat{A}$ project to symmetric and antisymmetric states. This means that for an arbitrary $|\psi\rangle \in V^{\otimes N}$, we find $\hat{S}|\psi\rangle \in \mathrm{Sym}^N V$.

$$\hat{P}_\alpha \hat{S}|\psi\rangle = \hat{S}|\psi\rangle, \quad \forall \alpha. \tag{12}$$

Analogously, for an arbitrary $|\psi\rangle \in V^{\otimes N}$ we find $\hat{A}|\psi\rangle \in \mathrm{Anti}^N V$.

$$\hat{P}_\alpha \hat{A}|\psi\rangle = \epsilon_\alpha \hat{A}|\psi\rangle, \quad \forall \alpha. \tag{13}$$

Hence, as claimed, $\hat{S}$ and $\hat{A}$ are projectors into the symmetric and antisymmetric subspaces:

$$\hat{S} : V^{\otimes N} \to \mathrm{Sym}^N V, \quad \hat{A} : V^{\otimes N} \to \mathrm{Anti}^N V. \tag{14}$$

Example: Symmetrizer and Antisymmetrizer for S_3

As we have seen before, the symmetric group S_3 has six permutation operators:

$$\hat{P}_{123} = 1, \quad \hat{P}_{312}, \quad \hat{P}_{231}, \quad \hat{P}_{132}, \quad \hat{P}_{213}, \quad \text{and} \quad \hat{P}_{321},$$

where the last three are transpositions. In this case the symmetrizer $\hat{S}$ and antisymmetrizer $\hat{A}$ are

$$\hat{S} = \frac{1}{6}(1 + \hat{P}_{312} + \hat{P}_{231} + \hat{P}_{132} + \hat{P}_{213} + \hat{P}_{321}), \tag{15}$$

$$\hat{A} = \frac{1}{6}(1 + \hat{P}_{312} + \hat{P}_{231} - \hat{P}_{132} - \hat{P}_{213} - \hat{P}_{321}). \tag{16}$$

For $N = 2$ the operators $\hat{S}$ and $\hat{A}$ add up to the identity, this does not hold for $N = 3$:

$$\hat{S} + \hat{A} = \frac{1}{3}(1 + \hat{P}_{312} + \hat{P}_{231}) \neq 1. \tag{17}$$

In fact, for $N > 2$ the direct sum of the purely symmetric and purely antisymmetric subspaces is a proper subspace of $V^{\otimes N}$:

$$\mathrm{Sym}^N V \oplus \mathrm{Anti}^N V \subset V^{\otimes N}, \quad N > 2. \tag{18}$$

For $N > 2$, the state space $V^{\otimes N}$ is not spanned by purely symmetric and purely antisymmetric states.

The Symmetrization Postulate

The Symmetrization Postulate

We will state the **symmetrization postulate**, which should be considered an assumption of quantum mechanics.

Symmetrization postulate:

> *In a system with N identical particles, the states that are physically realized are not arbitrary states in $V^{\otimes N}$ but rather are totally symmetric (that is, belonging to $\mathrm{Sym}^N V$), in which case the particles are said to be bosons, or they are totally antisymmetric (that is, belonging to $\mathrm{Anti}^N V$), in which case they are said to be fermions.*

It is intuitively clear that the exchange degeneracy problem has been solved. If we have two identical particles, one in state $|a\rangle$ and the other in a different state $|b\rangle$, the state of the two particles is neither $|a\rangle \otimes |b\rangle$ nor $|b\rangle \otimes |a\rangle$—the degenerate possibilities—but rather $|a\rangle \otimes |b\rangle \pm |b\rangle \otimes |a\rangle$, up to scale.

Symmetrization Postulate

- Let $|u\rangle \in V^{\otimes N}$ be an arbitrary vector in the tensor product. Associated with $|u\rangle$, we introduce the vector subspace $V_{|u\rangle}$ generated by acting on $|u\rangle$ with all the permutation operators in S_N:
$$V_{|u\rangle} \equiv \text{span}\{\hat{P}_\alpha \,|u\rangle \,,\ \forall\alpha\} \subset V^{\otimes N}. \tag{19}$$
E.g. for $N = 3$, $|u\rangle = |a\rangle_1 \,|a\rangle_2 \,|b\rangle_3$ and $V_{|u\rangle} = \text{span}\{|aab\rangle \,, |baa\rangle \,, |aba\rangle\}$.
- Depending on the choice of the state $|u\rangle$, the dimension of $V_{|u\rangle}$ can go from one to $N!$. This dimensionality, if different from one, is the degeneracy due to exchange.
- The exchange degeneracy problem is the ambiguity we face in selecting a representative for the physical state in $V_{|u\rangle}$. The problem is solved by the symmetrization postulate if we can show the following:

> **Claim:** Up to a multiplicative constant, $V_{|u\rangle}$ contains at most a single state in $\text{Sym}^N V$ and at most a single state in $\text{Anti}^N V$.

Proof of the Claim

Proof. We first show that, up to a multiplicative constant, $V_{|u\rangle}$ contains at most a single ket in $\mathrm{Sym}^N V$. Suppose we have a state $|\psi\rangle \in V_{|u\rangle}$ that is symmetric: $|\psi\rangle \in \mathrm{Sym}^N V$. Since $|\psi\rangle \in V_{|u\rangle}$, we can write it as follows:

$$|\psi\rangle = \sum_\alpha c_\alpha \hat{P}_\alpha |u\rangle \,, \tag{20}$$

with c_α some coefficients. Since $|\psi\rangle \in \mathrm{Sym}^N V$, it is left invariant by the action of $\hat{S}$:

$$|\psi\rangle = \hat{S}|\psi\rangle = \hat{S}\sum_\alpha c_\alpha \hat{P}_\alpha |u\rangle = \sum_\alpha c_\alpha \hat{S}\hat{P}_\alpha |u\rangle = \sum_\alpha c_\alpha \hat{S}|u\rangle = \hat{S}|u\rangle \sum_\alpha c_\alpha . \tag{21}$$

This shows that any symmetric $|\psi\rangle$ in $V_{|u\rangle}$ must be proportional to $\hat{S}|u\rangle$ and is therefore unique up to a multiplicative constant.

Proof of the Claim (Continued)

The argument is similar for the antisymmetric states. Suppose we have a state $|\psi\rangle \in V_{|u\rangle}$ that is antisymmetric: $|\psi\rangle \in \mathrm{Anti}^N V$. Again, we write it as

$$|\psi\rangle = \sum_\alpha d_\alpha \hat{P}_\alpha \, |u\rangle \, , \tag{22}$$

with d_α some coefficients. Since $|\psi\rangle \in \mathrm{Anti}^N V$, it is left invariant by the action of $\hat{A}$:

$$|\psi\rangle = \hat{A}|\psi\rangle = \hat{A}\sum_\alpha d_\alpha \hat{P}_\alpha |u\rangle = \sum_\alpha d_\alpha \hat{A}\hat{P}_\alpha |u\rangle = \sum_\alpha \epsilon_\alpha d_\alpha \hat{A}|u\rangle = \hat{A}|u\rangle \sum_\alpha \epsilon_\alpha d_\alpha . \tag{23}$$

This shows that any $|\psi\rangle$ must be proportional to $\hat{A}\,|u\rangle$ and is therefore unique up to a multiplicative constant.

Pauli's Exclusion Principle

The state in $\text{Anti}^N V$ will fail to exist in $V_{|u\rangle}$ if two or more V states appearing in $|u\rangle = |u_{i_1}\rangle \otimes \cdots \otimes |u_{i_N}\rangle$ are the same. This is the content of **Pauli's exclusion principle**, which states that two or more identical fermions cannot be found in the same state.

This is easily confirmed. Assume $|u\rangle = |u_{i_1}\rangle \otimes \cdots \otimes |u_{i_N}\rangle$ is such that two fermions, the pth and the qth, are in the same state $|u_k\rangle$. It then follows that the transposition (pq) leaves $|u\rangle$ invariant:

$$(pq)\,|u\rangle = |u\rangle\,.$$

Acting on this relation with the antisymmetrizer, we find

$$\hat{A}(pq)\,|u\rangle = \hat{A}\,|u\rangle\,.$$

Since (pq) is an odd permutation, this gives

$$-\hat{A}\,|u\rangle = \hat{A}\,|u\rangle\,,$$

resulting in $\hat{A}\,|u\rangle = 0$. Therefore, no state exists.

Additional Remarks on the Symmetrization Postulate (Part 1)

- **Two types of statistics:** The postulate describes particles with two kinds of statistics: bosons and fermions. Due to the constraint on their wave functions, the statistical behavior of bosons and fermions is significantly different.
- **Spin-statistics theorem:** Quantum field theory and special relativity together prove the spin-statistics theorem. It establishes that:
  - Bosons are particles of integer spin (0, 1, 2, ...).
  - Fermions are particles of half-integer spin (1/2, 3/2, ...).
- **Composite particles:** The symmetrization postulate for elementary particles implies a specific statistic for composite particles. Composite particles are either bosons or fermions, obeying the symmetrization postulate.

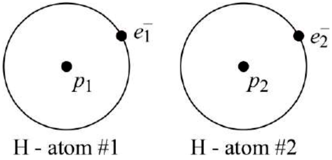


Figure: p_1 and e_1^- are the proton and electron of the first atom. p_2 and e_2^- are the proton and electron of the second atom.

Additional Remarks on the Symmetrization Postulate (Part 2)

Example: Two hydrogen atoms The two-atom system has four particles, with wave function $\psi(p_1, e_1^-; p_2, e_2^-)$. Since the two electrons are identical spin-$\frac{1}{2}$ particles, the wave function must be antisymmetric under the exchange $e_1^- \leftrightarrow e_2^-$:

$$\psi(p_1, e_2^-; p_2, e_1^-) = -\psi(p_1, e_1^-; p_2, e_2^-)$$

The same argument applies to the protons:

$$\psi(p_2, e_1^-; p_1, e_2^-) = -\psi(p_1, e_1^-; p_2, e_2^-)$$

Thus, under the simultaneous exchange of electrons and protons:

$$\psi(p_2, e_2^-; p_1, e_1^-) = \psi(p_1, e_1^-; p_2, e_2^-)$$

This exchange swaps the two hydrogen atoms. Since the wave function is symmetric, the hydrogen atom is a boson!

Additional Remarks on the Symmetrization Postulate (Part 3)

- The postulate is in fact a description of the most general statistics of particles that live in three spatial dimensions.
- Permutation-induced exchanges: the particles move in some trajectories to realize the permutations.
- In three spatial dimensions there are no new possibilities, just bosons or fermions.
- When space is two-dimensional, there are new possibilities. Transpositions of particles need not produce only sign factors on the wave function; general phases are in principle possible. Particles with such behavior are called anyons.

Operators transform under permutations

Consider an operator $\hat{B}(n)$ acting on the n-th vector space, such that:

$$\hat{B}(1)|u_i\rangle_{(1)} \otimes |u_j\rangle_{(2)} = \left(\hat{B}|u_i\rangle\right)_{(1)} \otimes |u_j\rangle_{(2)},$$

$$\hat{B}(2)|u_i\rangle_{(1)} \otimes |u_j\rangle_{(2)} = |u_i\rangle_{(1)} \otimes \left(\hat{B}|u_j\rangle\right)_{(2)}.$$

Action of $\hat{P}_{21}$ on $\hat{B}(1)$: Applying $\hat{P}_{21}$ on $\hat{B}(1)$, we have:

$$\hat{P}_{21}\hat{B}(1)\hat{P}_{21}|u_i\rangle_{(1)} \otimes |u_j\rangle_{(2)} = \hat{P}_{21}\left(B|u_j\rangle\right)_{(1)} \otimes |u_i\rangle_{(2)} = \hat{B}(2)|u_i\rangle \otimes |u_j\rangle.$$

Hence,

$$\hat{P}_{21}\hat{B}(1)\hat{P}_{21} = \hat{B}(2).$$

Similarly, we find:

$$\hat{P}_{21}\hat{B}(2)\hat{P}_{21} = \hat{B}(1).$$

Operators transform under permutations (continued)

General Operator Action: For a generic operator $\hat{\Theta}(1,2)$, we find:

$$\hat{P}_{21}\hat{\Theta}(1,2)\hat{P}_{21} = \hat{\Theta}(2,1).$$

If $\hat{\Theta}(1,2) = \hat{\Theta}(2,1)$, we say $\hat{\Theta}$ is symmetric. For a symmetric operator:

$$0 = \hat{P}_{21}\hat{\Theta}(1,2)\hat{P}_{21} - \hat{\Theta}(1,2),$$

which implies:

$$[\hat{P}_{21}, \hat{\Theta}(1,2)] = 0.$$

Thus,

$$[\hat{P}_{21}, \hat{\Theta}(1,2)] = 0 \iff \hat{\Theta} \text{ is symmetric.}$$

Operators transform under permutations

Any operator $\hat{B} \in \mathcal{L}(V)$, acting on the single-particle state space V, can define operators on the tensor product $V^{\otimes N}$, relevant for a system of N identical particles. We define $\hat{B}(k) \in \mathcal{L}(V^{\otimes N})$ as the operator that acts on the k-th state in any basis vector of $V^{\otimes N}$.
Examples include:

$$\hat{B}(1) = \hat{B} \otimes 1 \cdots \otimes 1, \quad \hat{B}(2) = 1 \otimes \hat{B} \otimes \cdots \otimes 1$$

Permutations act by conjugation on the $\hat{B}(k)$ operators and modify the state space they act upon. We claim that:

$$\hat{P}_\alpha^\dagger \hat{B}(k) \hat{P}_\alpha = \hat{B}(\alpha(k)) \tag{24}$$

Proof:

- $\hat{B}(k)$ acts on the state that $\hat{P}_\alpha$ places on the k-th position, specifically, $|u_{\alpha(k)}\rangle$.
- Finally, $\hat{P}_\alpha^\dagger = (\hat{P}_\alpha)^{-1}$ brings back this state, with $\hat{B}$ acting on it, to its original position.

Symmetric Operators and Permutations

Consider a general operator $\hat{\Theta}(1, \ldots, N)$, constructed from an arbitrary set of operators acting on various particles. Let $1, \ldots, N$ be the labels of these operators, specifying the state space on which they act.
From Eq. (24), we have:

$$\hat{P}_\alpha^\dagger \hat{\Theta}(1, \ldots, N) \hat{P}_\alpha = \hat{\Theta}(\alpha(1), \ldots, \alpha(N)),$$

where the arguments on the right side are a reordering of $1, \ldots, N$. A Hermitian operator $\hat{M}(1, 2, \ldots, N)$ is completely symmetric if, for any permutation α, it satisfies:

$$\hat{M}(\alpha(1), \ldots, \alpha(N)) = \hat{M}(1, 2, \ldots, N).$$

For such operators, we have:

$$\hat{P}_\alpha^\dagger \hat{M}(1, 2, \ldots, N) \hat{P}_\alpha = \hat{M}(1, 2, \ldots, N).$$

Conservation and Symmetrization

A completely symmetric operator commutes with all permutation operators:

$$[\hat{M}(1, 2, \ldots, N), \hat{P}_\alpha] = 0, \quad \forall \alpha.$$

For identical particles, the Hamiltonian $\hat{H}(1, \ldots, N)$ must be a completely symmetric observable, implying:

$$[\hat{H}(1, \ldots, N), \hat{P}_\alpha] = 0.$$

This leads to the conservation of all permutation operators $\hat{P}_\alpha$, giving:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \hat{P}_\alpha \rangle = \frac{i}{\hbar}\left\langle [\hat{H}, \hat{P}_\alpha] \right\rangle = 0.$$

Both symmetrizer $\hat{S}$ and antisymmetrizer $\hat{A}$, being sums of permutation operators, are conserved:

$$[\hat{H}(1, \ldots, N), \hat{S}] = [\hat{H}(1, \ldots, N), \hat{A}] = 0.$$

Time Evolution of Symmetric and Antisymmetric States

The conservation of $\hat{S}$ and $\hat{A}$ implies they commute with the unitary time evolution operator $\mathcal{U}$. Thus:

- If a state is fully symmetric at $t = 0$, it remains symmetric for all times.
- If a state is totally antisymmetric at $t = 0$, it remains antisymmetric for all times.

This ensures that the symmetry properties of the initial state are preserved throughout time evolution.

Probabilities

Building Antisymmetric States

Suppose you want to build a three-fermion state starting from the following state $|u\rangle \in V^{\otimes 3}$:

$$|u\rangle = |\omega_1\rangle_{(1)} \otimes |\omega_2\rangle_{(2)} \otimes |\omega_3\rangle_{(3)} \,.$$

The antisymmetrized state, obtained by acting with the antisymmetrizer on $|u\rangle$, is:

$$\begin{aligned}
\hat{A}\,|u\rangle &= \frac{1}{3!}\sum_{\alpha} \epsilon_\alpha \hat{P}_\alpha \, |\omega_1\rangle_{(1)} \otimes |\omega_2\rangle_{(2)} \otimes |\omega_3\rangle_{(3)} \\
&= \frac{1}{3!}\begin{vmatrix} |\omega_1\rangle_{(1)} & |\omega_1\rangle_{(2)} & |\omega_1\rangle_{(3)} \\ |\omega_2\rangle_{(1)} & |\omega_2\rangle_{(2)} & |\omega_2\rangle_{(3)} \\ |\omega_3\rangle_{(1)} & |\omega_3\rangle_{(2)} & |\omega_3\rangle_{(3)} \end{vmatrix} = \frac{1}{3!}\epsilon_{ijk}\, |\omega_i\rangle_{(1)} \otimes |\omega_j\rangle_{(2)} \otimes |\omega_k\rangle_{(3)} \,.
\end{aligned}$$

Here, repeated indices are summed over. ϵ_{ijk} is the parity of the permutation $\hat{P}_{ijk}$.

Slater Determinant

Recall the general formula for the determinant of an $N \times N$ matrix B_{ij}:

$$\det B = \sum_{\alpha} \epsilon_\alpha B_{\alpha(1),1} B_{\alpha(2),2} \ldots B_{\alpha(N),N}.$$

Now, let $|\omega\rangle \in V^{\otimes N}$ be a state of the form

$$|\omega\rangle = |\omega_1\rangle_{(1)} \otimes |\omega_2\rangle_{(2)} \otimes \cdots \otimes |\omega_N\rangle_{(N)},$$

with $|\omega_i\rangle \in V$ for $i = 1, \ldots, N$. We define the N-fermion state $|\psi_\omega\rangle$ up to normalization as:

$$|\psi_\omega\rangle = \hat{A}\,|\omega\rangle,$$

where $\hat{A}$ is the antisymmetrizer. To calculate $\hat{A}\,|\omega\rangle$, note that the action of the permutation $\hat{P}_\alpha$ on $|\omega\rangle$ gives:

$$\hat{P}_\alpha\,|\omega\rangle = |\omega_{\alpha(1)}\rangle_{(1)} \otimes |\omega_{\alpha(2)}\rangle_{(2)} \otimes \cdots \otimes |\omega_{\alpha(N)}\rangle_{(N)}.$$

Therefore, applying the antisymmetrizer yields:

$$|\psi_\omega\rangle = \hat{A}\,|\omega\rangle = \frac{1}{N!}\sum_{\alpha} \epsilon_\alpha \hat{P}_\alpha\,|\omega\rangle = \frac{1}{N!}\sum_{\alpha} \epsilon_\alpha\,|\omega_{\alpha(1)}\rangle_{(1)} \otimes \cdots \otimes |\omega_{\alpha(N)}\rangle_{(N)}.$$

Constructing the N-Fermion State

Define a matrix ω_{ij} of kets, where the row index i labels the different states and the column index j labels the various particles:

$$\omega_{ij} \equiv |\omega_i\rangle_{(j)} .$$

With this definition, the antisymmetrized state $|\psi_\omega\rangle$ becomes:

$$\begin{aligned} |\psi_\omega\rangle = \hat{A}\,|\omega\rangle = \frac{1}{N!}\sum_\alpha \epsilon_\alpha\,\omega_{\alpha(1),1}\cdots\omega_{\alpha(N),N} = \frac{1}{N!}\det\omega. \\ = \frac{1}{N!}\begin{vmatrix} |\omega_1\rangle_{(1)} & \cdots & |\omega_1\rangle_{(N)} \\ \vdots & & \vdots \\ |\omega_N\rangle_{(1)} & \cdots & |\omega_N\rangle_{(N)} \end{vmatrix} \end{aligned} \tag{25}$$

This final expression for $|\psi_\omega\rangle$ is known as the **Slater determinant** for the set of states $\{|\omega_i\rangle\}$.

Normalization

We can use the Slater determinant to construct the position-space wave function for a system of N identical fermions by applying the position-space bra on both sides of equation (25)

$$\psi_\omega(\vec{x}_1, \ldots, \vec{x}_N) \equiv \langle \vec{x}_1, \ldots, \vec{x}_N | \psi_\omega \rangle,$$

with $\langle \vec{x}_1, \ldots, \vec{x}_N | \equiv {}_{(1)} \langle \vec{x}_1 | \otimes \cdots \otimes_{(N)} \langle \vec{x}_N |$. On the right-hand side, bringing in the bras with particle labels, we get

$$\psi_\omega(\vec{x}_1, \ldots, \vec{x}_N) = \mathcal{N} \det \begin{vmatrix} \omega_1(\vec{x}_1) & \cdots & \omega_1(\vec{x}_N) \\ \vdots & \ddots & \vdots \\ \omega_N(\vec{x}_1) & \cdots & \omega_N(\vec{x}_N) \end{vmatrix}. \tag{26}$$

with $\mathcal{N}$ some normalization constant.

Properties and Implications of the Fermionic Wave Function

The fermionic wave function $\psi_\omega(\vec{x}_1, \ldots, \vec{x}_N)$ defined by the Slater determinant has key properties:

- **Pauli's Exclusion Principle:** If $\omega_i(\vec{x}) = \omega_j(\vec{x})$ for any $i \neq j$, the determinant vanishes:

$$\psi_\omega(\vec{x}_1, \ldots, \vec{x}_N) = 0.$$

- **Antisymmetry under Exchange:** Swapping two coordinates $\vec{x}_i$ and $\vec{x}_j$ changes the sign of the wave function, enforcing fermionic statistics.

This determinant form, or Slater determinant, is foundational for describing N-fermion systems in quantum mechanics.

Normalizing Wave Functions and Probabilities

- We have already defined position-space bras: $\langle \vec{x}_1, \ldots, \vec{x}_N| = {}_{(1)}\langle \vec{x}_1| \otimes \cdots \otimes {}_{(N)}\langle \vec{x}_N|$. The overlap of such states with similarly defined kets is given by:

$$\langle \vec{x}_1, \ldots, \vec{x}_N | \vec{y}_1, \ldots, \vec{y}_N \rangle = \delta^{(3)}(\vec{x}_1 - \vec{y}_1) \cdots \delta^{(3)}(\vec{x}_N - \vec{y}_N).$$

- The completeness relation in the N-particle state space takes the form

$$\int \mathrm{d}^3\vec{x}_1 \cdots \mathrm{d}^3\vec{x}_N \, |\vec{x}_1, \ldots, \vec{x}_N\rangle \, \langle \vec{x}_1, \ldots, \vec{x}_N| = 1.$$

- Here, each integral is three-dimensional and runs over all of space. This completeness can be verified by acting on $|\vec{y}_1, \ldots, \vec{y}_N\rangle$.
- We define the position-space wave function $\psi(\vec{x}_1, \ldots, \vec{x}_N)$ associated with an N-particle state $|\psi\rangle \in V^{\otimes N}$: $\psi(\vec{x}_1, \ldots, \vec{x}_N) \equiv \langle \vec{x}_1, \ldots, \vec{x}_N | \psi \rangle$.
- The normalization condition for the wave function is:

$$\int \mathrm{d}^3\vec{x}_1 \cdots \mathrm{d}^3\vec{x}_N |\psi(\vec{x}_1, \ldots, \vec{x}_N)|^2 = 1.$$

Note: $|\psi(\vec{x}_1, \ldots, \vec{x}_N)|^2$ is a totally symmetric function of the arguments—this is true for identical bosons and for identical fermions because any exchange of arguments changes the wave function only by a sign.

Probability for Two Identical Particles

- Consider, for example, the case of two identical particles and the expression

$$\mathrm{d}^3\vec{x}_1\mathrm{d}^3\vec{x}_2\,|\psi(\vec{x}_1,\vec{x}_2)|^2.$$

- If the particles are distinguishable, this gives the probability of finding the first particle at $\vec{x}_1$ in $\mathrm{d}^3\vec{x}_1$ and the second particle at $\vec{x}_2$ in $\mathrm{d}^3\vec{x}_2$.
- For identical particles, we should ask: *What is the probability of finding one particle at $\vec{x}_A$ within $\mathrm{d}^3\vec{x}_A$ and the other at $\vec{x}_B$ within $\mathrm{d}^3\vec{x}_B$?* Using the double integral:

$$\int \mathrm{d}^3\vec{x}_1\mathrm{d}^3\vec{x}_2\,|\psi(\vec{x}_1,\vec{x}_2)|^2 = 1,$$

the configuration $(\vec{x}_1,\vec{x}_2) = (\vec{x}_A,\vec{x}_B)$ or $(\vec{x}_1,\vec{x}_2) = (\vec{x}_B,\vec{x}_A)$ both contribute equally, leading to the probability:

$$\text{Probability of finding a particle at } \vec{x}_A \text{ and another at } \vec{x}_B = 2\cdot \mathrm{d}^3\vec{x}_A\mathrm{d}^3\vec{x}_B\,|\psi(\vec{x}_A,\vec{x}_B)|^2.$$

Generalization for N-Particle States

More generally, the probability of finding in an N-particle state a particle at each of the positions $(\vec{x}_1, \cdots, \vec{x}_N)$ with ranges $(\mathrm{d}^3\vec{x}_1, \cdots, \mathrm{d}^3\vec{x}_N)$ is:

$$N!\,\mathrm{d}^3\vec{x}_1 \cdots \mathrm{d}^3\vec{x}_N\,|\psi(\vec{x}_1, \ldots, \vec{x}_N)|^2. \tag{27}$$

However, the probability of finding all identical particles at the same position $\vec{x}$ within $\mathrm{d}^3\vec{x}$ does **not** include the extra $N!$ factor:

$$\text{Probability of all particles at } \vec{x} = \mathrm{d}^3\vec{x} \cdots \mathrm{d}^3\vec{x}\,|\psi(\vec{x}, \ldots, \vec{x})|^2.$$

Normalization of the N-Fermion Wave Function

Let us return to address the normalization of the N-fermion wave function (26). This cannot be done in general unless we know the orthogonality properties of the one-particle wave functions. Assume the $\omega_1(\vec{x}), \ldots, \omega_N(\vec{x})$ are orthonormal:

$$\int \mathrm{d}^3\vec{x}\, \omega_i^*(\vec{x})\, \omega_j(\vec{x}) = \delta_{ij}, \quad i, j = 1, \ldots, N.$$

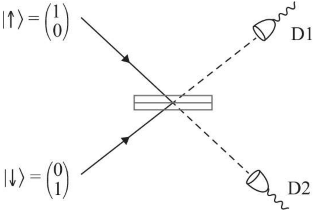
In this case, we achieve proper normalization with

$$\psi(\vec{x}_1, \ldots, \vec{x}_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \omega_1(\vec{x}_1) & \ldots & \omega_1(\vec{x}_N) \\ \vdots & \ddots & \vdots \\ \omega_N(\vec{x}_1) & \ldots & \omega_N(\vec{x}_N) \end{vmatrix}.$$

This is clear: The determinant is the sum of $N!$ terms. In the calculation of $\int \mathrm{d}^3\vec{x}_1 \cdots \mathrm{d}^3\vec{x}_N |\psi|^2$, due to orthonormality, each term from the determinant gives a $+1$ contribution when integrated against its complex conjugate, and zero otherwise. The $N!$ terms yield a total of $N!$ that is canceled precisely by the square of the determinant prefactor in the expression. Thus, ψ satisfies $\int \mathrm{d}^3\vec{x}_1 \cdots \mathrm{d}^3\vec{x}_N |\psi|^2 = 1$, as required.

Hong-Ou-Mandel Two-Photon Experiment

Consider the experiment where two identical photons hit a balanced beam splitter simultaneously. The claim was that both photons must end in one detector or the other; the processes in which one photon ends in each detector interfere destructively.



Two identical photons hit a balanced beam splitter (the reflection and transmission probabilities are the same), one from the top and one from the bottom. The D1 and D2 detectors never click simultaneously; the two photons both end up together on either D1 or D2.

The action of a beam splitter with two input ports, top and bottom, can be represented by a 2×2 unitary matrix U of the form

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad |\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have defined the state $|\uparrow\rangle$, representing a photon incident from the top, and the state $|\downarrow\rangle$, representing a photon incident from the bottom.

Hong-Ou-Mandel Two-Photon Experiment

- To the right of the beam splitter, the state $|\uparrow\rangle$ is a photon heading to D1, while the state $|\downarrow\rangle$ is a photon heading to D2.
- The incoming state of two photons, one incident from the top and one incident from the bottom, must be symmetrized because the particles are identical bosons. The normalized incident state $|\Psi_{\text{inc}}\rangle$ is therefore

$$|\Psi_{\text{inc}}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_{(1)} \otimes |\downarrow\rangle_{(2)} + |\downarrow\rangle_{(1)} \otimes |\uparrow\rangle_{(2)}\right).$$

- Since each photon experiences the action of the beam splitter, the outgoing state $|\Psi_{\text{out}}\rangle$ is

$$\begin{aligned}|\Psi_{\text{out}}\rangle = (U \otimes U)\,|\Psi_{\text{inc}}\rangle &= \frac{1}{\sqrt{2}}\Big[(U\,|\uparrow\rangle)_{(1)} \otimes (U\,|\downarrow\rangle)_{(2)} + (U\,|\downarrow\rangle)_{(1)} \otimes (U\,|\uparrow\rangle)_{(2)}\Big] \\ &= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_{(1)} \otimes |\uparrow\rangle_{(2)} - |\downarrow\rangle_{(1)} \otimes |\downarrow\rangle_{(2)}\right)\end{aligned}$$

  The first term gives the amplitude for both photons to end up in D1, and the second term gives the amplitude for both photons to end up in D2. As claimed, we can't get a single photon in each detector.

Two Sets of Degrees of Freedom

Particles with Two Sets of Degrees of Freedom

Consider a particle that has both **spatial** degrees of freedom and **spin** degrees of freedom.

$$|\psi_{\text{spatial}}\rangle \otimes |\psi_{\text{spin}}\rangle \,.$$

Let V be the vector space of spatial states, and let W be the vector space of spin states. The general states $|\psi\rangle$ of the particle belong to the tensor product:

$$|\psi\rangle \in V \otimes W.$$

The space $V \otimes W$ is the state space of the particle. If we had N such particles, the states of the system would live in the Nth tensor product of the single-particle state space:

$$(V \otimes W)^{\otimes N} = (V \otimes W) \cdots (V \otimes W).$$

If you think of basis elements of tensor products as ordered lists of states, it is clear that one can identify the following spaces:

$$(V \otimes W)^{\otimes N} \cong V^{\otimes N} \otimes W^{\otimes N}.$$

On the right-hand side, we are listing first the N basis vectors in V and then the N basis vectors in W. For the case of two identical particles, $N = 2$, the above identification reads

$$(V \otimes W)^{\otimes 2} \cong V^{\otimes 2} \otimes W^{\otimes 2}.$$

Exchange of Particles in $V \otimes W$ Space

Now consider a state of two particles and the identification above::

$$\Big(|v_1\rangle_{(1)} \otimes |w_1\rangle_{(1)}\Big) \otimes \Big(|v_2\rangle_{(2)} \otimes |w_2\rangle_{(2)}\Big) \cong \Big(|v_1\rangle_{(1)} \otimes |v_2\rangle_{(2)}\Big) \otimes \Big(|w_1\rangle_{(1)} \otimes |w_2\rangle_{(2)}\Big) \,.$$

To explore particle exchange, we focus on the state on the right. Treating it as a list of four states, we use the language of permutations. Here, exchanging the two particles is represented by $\hat{P}_{2143}$:

$$\hat{P}_{2143}\,|v_1\rangle_{(1)} \otimes |v_2\rangle_{(2)} \otimes |w_1\rangle_{(1)} \otimes |w_2\rangle_{(2)} = |v_2\rangle_{(1)} \otimes |v_1\rangle_{(2)} \otimes |w_2\rangle_{(1)} \otimes |w_1\rangle_{(2)} \,.$$

Eigenvalues of Permutation Operators

- This permutation exchanges both V and W states. We can express $\hat{P}_{2143}$ as a product of two independent permutations:

$$\hat{P}_{2143} = \hat{P}_{2134} \cdot \hat{P}_{1243}.$$

  $\hat{P}_{2134}$ permutes the V states, while $\hat{P}_{1243}$ permutes the W states. These operators commute, each squaring to the identity, and each with eigenvalues ± 1. Hence, they can be diagonalized simultaneously. They also commute with $\hat{P}_{2143}$, so diagonalizing $\hat{P}_{2134}$ and $\hat{P}_{1243}$ simultaneously diagonalizes $\hat{P}_{2143}$ as well.
- Denoting the eigenvalues by λ, we have:

$$\lambda_{2143} = \lambda_{2134}\lambda_{1243}.$$

  The possible values of λ are summarized in the table below:

λ_{2143}	λ_{2134}	λ_{1243}
1	1	1
1	-1	-1
-1	1	-1
-1	-1	1

Symmetrization and Antisymmetrization of $V \otimes W$ States

Since $\hat{P}_{2143}$ is the operator that exchanges the two particles, the states with $\lambda_{2143} = 1$ live in $\mathrm{Sym}^2(V \otimes W)$ (bosons), and the states with $\lambda_{2143} = -1$ live in $\mathrm{Anti}^2(V \otimes W)$ (fermions). The value $\lambda_{2143} = 1$ is realized in two ways:

$$\lambda_{2134} = \lambda_{1243} = 1, \quad \text{or} \quad \lambda_{2134} = \lambda_{1243} = -1.$$

The states for $\lambda_{2134} = 1$ are in $\mathrm{Sym}^2 V$, and those for $\lambda_{1243} = 1$ are in $\mathrm{Sym}^2 W$, implying:

$$\lambda_{2134} = \lambda_{1243} = 1 \Rightarrow \mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 W.$$

Similarly, the states for which $\lambda_{2134} = -1$ are in $\mathrm{Anti}^2 V$, and those for $\lambda_{1243} = -1$ are in $\mathrm{Anti}^2 W$, resulting in:

$$\lambda_{2134} = \lambda_{1243} = -1 \Rightarrow \mathrm{Anti}^2 V \otimes \mathrm{Anti}^2 W.$$

Decomposition of Symmetric and Antisymmetric Spaces

The states with $\lambda_{2143} = -1$ belong to $\mathrm{Anti}^2(V \otimes W)$ and can arise in two ways:

$$\lambda_{2134} = -\lambda_{1243} = 1, \quad \text{or} \quad -\lambda_{2134} = \lambda_{1243} = 1.$$

This results in the decomposition:

$$\mathrm{Anti}^2(V \otimes W) \cong (\mathrm{Sym}^2 V \otimes \mathrm{Anti}^2 W) \oplus (\mathrm{Anti}^2 V \otimes \mathrm{Sym}^2 W).$$

Summarizing, we obtain:

$$\mathrm{Sym}^2(V \otimes W) \cong (\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 W) \oplus (\mathrm{Anti}^2 V \otimes \mathrm{Anti}^2 W),$$

$$\mathrm{Anti}^2(V \otimes W) \cong (\mathrm{Sym}^2 V \otimes \mathrm{Anti}^2 W) \oplus (\mathrm{Anti}^2 V \otimes \mathrm{Sym}^2 W).$$

States of Two-Electron Systems

States of Two-Electron Systems

- Consider the wave function ψ for two electrons. This wave function depends on the position of each electron and the spin state of each electron.

$$\psi\left(\mathbf{x}_1, m_1; \mathbf{x}_2, m_2\right) = \phi\left(\mathbf{x}_1, \mathbf{x}_2\right) \cdot \chi\left(m_1, m_2\right).$$

- Assume that χ is a normalized spin state. It can be viewed as a superposition of the triplet and the singlet states that arise by combining the two spins.
- Focus on two important alternatives, denoted as ψ_1 and ψ_2. The total wave function must be antisymmetric under the exchange of the electrons,

$$\psi_1\left(\mathbf{x}_1, m_1; \mathbf{x}_2, m_2\right) = \phi_+\left(\mathbf{x}_1, \mathbf{x}_2\right) \cdot \chi_{\text{singlet}}\left(m_1, m_2\right)$$
$$\psi_2\left(\mathbf{x}_1, m_1; \mathbf{x}_2, m_2\right) = \phi_-\left(\mathbf{x}_1, \mathbf{x}_2\right) \cdot \chi_{\text{triplet}}\left(m_1, m_2\right).$$

- Since the singlet is antisymmetric and the triplet is symmetric, the spatial wave functions must obey the following relations

$$\phi_\pm\left(\mathbf{x}_1, \mathbf{x}_2\right) = \pm\phi_\pm\left(\mathbf{x}_2, \mathbf{x}_1\right).$$

States of Two-Electron Systems (continued)

- Recalling (27), the probability of finding one electron at positions $\mathbf{x}_1$ and the other electron at $\mathbf{x}_2$ is
$$2\mathrm{d}P = 2\mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\left|\phi\left(\mathbf{x}_1,\mathbf{x}_2\right)\right|^2.$$
- For simplicity, we will assume that the electrons do not interact with each other and that the total Hamiltonian is spin independent
$$\hat{H}_{\mathrm{tot}} = \hat{H}\otimes 1 + 1\otimes\hat{H}, \quad \text{with} \quad \hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{x})$$
- The time-independent Schrödinger equation for the spatial part of the wave function is
$$\left[-\frac{\hbar^2}{2m}\nabla^2_{\mathbf{x}_1} + V\left(\mathbf{x}_1\right) - \frac{\hbar^2}{2m}\nabla^2_{\mathbf{x}_2} + V\left(\mathbf{x}_2\right)\right]\phi\left(\mathbf{x}_1,\mathbf{x}_2\right) = E\phi\left(\mathbf{x}_1,\mathbf{x}_2\right).$$
- This equation is separable, so there is a solution of the form $\phi\left(\mathbf{x}_1,\mathbf{x}_2\right) = \phi_A\left(\mathbf{x}_1\right)\phi_B\left(\mathbf{x}_2\right)$,
$$\left[-\frac{\hbar^2}{2m}\nabla^2_{\mathbf{x}} + V(\mathbf{x})\right]\phi_A(\mathbf{x}) = E_A\phi_A(\mathbf{x}), \quad \left[-\frac{\hbar^2}{2m}\nabla^2_{\mathbf{x}} + V(\mathbf{x})\right]\phi_B(\mathbf{x}) = E_B\phi_B(\mathbf{x}),$$
with $E = E_A + E_B$. We can choose the ϕ_A and ϕ_B wave functions to be normalized, but they need not be orthogonal.
$$\langle\phi_A,\phi_B\rangle = \int \mathrm{d}^3\mathbf{x}\,\phi_A^*(\mathbf{x})\phi_B(\mathbf{x}) \equiv \alpha_{AB} \neq 0.$$

Degenerate States and Normalization

Conditions for Degenerate States:

- States of a Hamiltonian with different energies are orthogonal.
- For a nonzero α_{AB}, ϕ_A and ϕ_B must be degenerate in the spectrum of $\hat{H}$, implying $E_A = E_B$.

By Schwarz's inequality,

$$|\langle \phi_A, \phi_B \rangle|^2 \leq \langle \phi_A, \phi_A \rangle \langle \phi_B, \phi_B \rangle = 1,$$

showing that $|\alpha_{AB}|^2 \leq 1$ or equivalently, $|\alpha_{AB}| \leq 1$.

Constructing Symmetric and Antisymmetric Spatial Wave Functions:

- Define spatial wave functions $\phi_\pm(\mathbf{x}_1, \mathbf{x}_2)$ with definite exchange symmetry as:

$$\phi_\pm(\mathbf{x}_1, \mathbf{x}_2) = \frac{N_\pm}{\sqrt{2}} [\phi_A(\mathbf{x}_1)\phi_B(\mathbf{x}_2) \pm \phi_A(\mathbf{x}_2)\phi_B(\mathbf{x}_1)] \tag{28}$$

where $N_\pm$ is a real normalization constant.

Normalization Condition

Writing out the probability, we find:

$$\mathrm{d}^3\mathbf{x}_1\mathrm{d}^3\mathbf{x}_2|\phi_\pm(\mathbf{x}_1,\mathbf{x}_2)|^2$$
$$= \frac{1}{2}N_\pm^2\mathrm{d}^3\mathbf{x}_1\mathrm{d}^3\mathbf{x}_2\left[|\phi_A(\mathbf{x}_1)|^2|\phi_B(\mathbf{x}_2)|^2+|\phi_A(\mathbf{x}_2)|^2|\phi_B(\mathbf{x}_1)|^2\pm 2\,\mathrm{Re}\,(\phi_A^*(\mathbf{x}_1)\phi_B^*(\mathbf{x}_2)\phi_A(\mathbf{x}_2)\phi_B(\mathbf{x}_1))\right]$$

Since the integral of the left-hand side must equal one, we have:

$$1 = \frac{1}{2}N_\pm^2\left(1+1\pm 2\,|\alpha_{AB}|^2\right) \quad \Rightarrow \quad N_\pm = \frac{1}{\sqrt{1\pm|\alpha_{AB}|^2}}.$$

Comparison with Orthogonal States:

- When ϕ_A and ϕ_B are orthogonal, normalization is reduced for the symmetric wave function and increased for the antisymmetric wave function.

Probability of Finding Electrons at the Same Position

Calculating $\mathrm{d}P_\pm$:

- The probability $\mathrm{d}P_\pm$ of finding both electrons within $\mathrm{d}^3\mathbf{x}$ of the same position $\mathbf{x}$ is given by:

$$\begin{aligned}\mathrm{d}P_\pm &= \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}|\phi_\pm(\mathbf{x},\mathbf{x})|^2 \\ &= \frac{1}{2}N_\pm^2\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}\left[2|\phi_A(\mathbf{x})|^2|\phi_B(\mathbf{x})|^2 \pm |\phi_A(\mathbf{x})|^2|\phi_B(\mathbf{x})|^2\right] = N_\pm^2 dxdx|\phi_A(\mathbf{x})|^2|\phi_B(\mathbf{x})|^2(1\pm 1)\end{aligned}$$

- Substituting the value of $N_\pm$ determined earlier, we obtain:

$$\mathrm{d}P_+ = \frac{2}{1+|\alpha_{AB}|^2}|\phi_A(\mathbf{x})|^2|\phi_B(\mathbf{x})|^2\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}, \quad \mathrm{d}P_- = 0.$$

Interpretation of $\mathrm{d}P_\pm$:

- $\mathrm{d}P_+$ corresponds to the spin singlet state, while $\mathrm{d}P_-$ is associated with the spin triplet.
- Electrons avoid each other in the triplet state due to the antisymmetry of the spatial wave function, which corresponds to $\mathrm{d}P_- = 0$.
- For symmetric spatial wave functions, $\mathrm{d}P_+$ is enhanced, indicating a higher probability of the particles being at the same point compared to distinguishable particles.

Comparison with Distinguishable Particles

Wave Function for Distinguishable Particles:

- For distinguishable particles, the normalized wave function $\phi_D(\mathbf{x}_1, \mathbf{x}_2)$ is given by:

$$\phi_D(\mathbf{x}_1, \mathbf{x}_2) = \phi_A(\mathbf{x}_1)\phi_B(\mathbf{x}_2),$$

  which represents the amplitude for the particle in state A to be at x_1 and the particle in state B to be at x_2.
- The probability $\mathrm{d}P_D$ of both particles being at the same point is:

$$\mathrm{d}P_D = |\phi_A(\mathbf{x})|^2 |\phi_B(\mathbf{x})|^2 \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{x}.$$

Relation to $\mathrm{d}P_+$:

- We find:

$$\mathrm{d}P_+ = \frac{2}{1 + |\alpha_{AB}|^2} \mathrm{d}P_D.$$

- This shows that the probability of finding identical particles at the same place is enhanced compared to distinguishable particles by a factor of two, reduced slightly by the overlap of one-particle states.

Thus, we have demonstrated the conditions under which identical particles exhibit enhanced probability at the same point, as desired. $\mathrm{d}P_+ \geq \mathrm{d}P_D$

Particles at Different Points

Probability of Identical Particles:

- More generally, for particles located at different points, we can consider the probability of finding one particle at $\mathbf{x}_1$ within $\mathrm{d}^3\mathbf{x}_1$, and the other at $\mathbf{x}_2$ within $\mathrm{d}^3\mathbf{x}_2$.
- For identical particles, this probability $\mathrm{d}P_{I,\pm}$ (with I indicating identical particles) is:

$$\mathrm{d}P_{I,\pm} = 2\,\mathrm{d}^3\mathbf{x}_1\mathrm{d}^3\mathbf{x}_2\,|\phi_\pm(\mathbf{x}_1,\mathbf{x}_2)|^2.$$

Probability of Distinguishable Particles:

- For distinguishable particles, we calculate the same quantity by summing probabilities:
  - The probability that particle A is at $\mathbf{x}_1$ and particle B at $\mathbf{x}_2$.
  - Plus, the probability that particle A is at $\mathbf{x}_2$ and particle B at $\mathbf{x}_1$.
- This total probability $\mathrm{d}P_{D,\mathrm{tot}}$ for distinguishable particles is:

$$\mathrm{d}P_{D,\mathrm{tot}} = \mathrm{d}^3\mathbf{x}_1\mathrm{d}^3\mathbf{x}_2\left(|\phi_A(\mathbf{x}_1)|^2|\phi_B(\mathbf{x}_2)|^2 + |\phi_A(\mathbf{x}_2)|^2|\phi_B(\mathbf{x}_1)|^2\right).$$

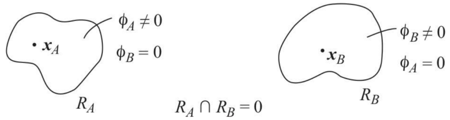
Comparison and Expectations:

- For the $+$ case, when $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$ and $\mathrm{d}^3\mathbf{x}_1 = \mathrm{d}^3\mathbf{x}_2 = \mathrm{d}^3\mathbf{x}$, we observe that $\mathrm{d}P_{I,+} = 2\mathrm{d}P_+$ and $\mathrm{d}P_{D,\text{tot}} = 2\mathrm{d}P_D$.
- This confirms that we are comparing the correct quantities.
- More generally, we expect $\mathrm{d}P_{I,+} > \mathrm{d}P_{D,\text{tot}}$ when $\mathbf{x}_1$ is near $\mathbf{x}_2$, a relationship that could be verified in specific examples.
- Additionally, we expect $\mathrm{d}P_{I,-} < \mathrm{d}P_{D,\text{tot}}$ for x_1 near x_2.

This concludes the probability comparison for finding identical and distinguishable particles at different points.

Non-Overlapping Wave Functions

Motivation: We explore why we don't symmetrize wave functions for distant electrons, like those in the lab and on the moon. When an electron's wave function is localized on the moon, it has no support in the lab, and vice versa. The question essentially addresses how to handle identical particles that are spatially separated with no wave function overlap.



Two particles localized in disjoint regions R_A and R_B. The one-particle wave function ϕ_A and ϕ_B have support in R_A and R_B, respectively.

Disjoint Support of Wave Functions: Consider two identical particles with wave functions localized in separate, non-overlapping regions R_A and R_B (illustrated in Figure).

- $\phi_A(\mathbf{x})$ is nonzero only in R_A, while $\phi_B(\mathbf{x})$ is nonzero only in R_B.
- The regions are disjoint, meaning $R_A \cap R_B = 0$.

Under these conditions, the overlap integral vanishes:

$$\alpha_{AB} = \int \mathrm{d}^3\mathbf{x}\, \phi_A(\mathbf{x})\phi_B(\mathbf{x}) = 0 \quad \Rightarrow \quad N_\pm = 1.$$

Because ϕ_A and ϕ_B have disjoint supports, ensuring no overlap at any point x.

Calculating Probability for Non-Overlapping Regions

Setup of Symmetrized Wave Functions: Using our symmetrized wave functions, we calculate the probability $\mathrm{d}P$ of finding an electron in $\mathrm{d}^3\mathbf{x}_A$ around $\mathbf{x}_A \in R_A$ and another in $\mathrm{d}^3\mathbf{x}_B$ around $\mathbf{x}_B \in R_B$.
The probability $\mathrm{d}P$ is given by:

$$\mathrm{d}P = \mathrm{d}^3\mathbf{x}_A \mathrm{d}^3\mathbf{x}_B \left(|\phi_\pm(\mathbf{x}_A, \mathbf{x}_B)|^2 + |\phi_\pm(\mathbf{x}_B, \mathbf{x}_A)|^2\right).$$

Form of $\phi_\pm(\mathbf{x}_A, \mathbf{x}_B)$: With normalization $N_\pm = 1$ (as shown by Eq. (28)), the symmetrized and antisymmetrized wave functions are:

$$\phi_\pm(\mathbf{x}_A, \mathbf{x}_B) = \frac{1}{\sqrt{2}}\left[\phi_A(\mathbf{x}_A)\phi_B(\mathbf{x}_B) \pm \phi_A(\mathbf{x}_B)\phi_B(\mathbf{x}_A)\right] = \frac{1}{\sqrt{2}}\phi_A(\mathbf{x}_A)\phi_B(\mathbf{x}_B).$$

When swapped, we have:

$$\phi_\pm(\mathbf{x}_B, \mathbf{x}_A) = \pm\frac{1}{\sqrt{2}}\phi_A(\mathbf{x}_A)\phi_B(\mathbf{x}_B).$$

Concluding Probability Expression for Distinguishable Particles

Simplifying the Probability Expression: Substituting $\phi_\pm(\mathbf{x}_A, \mathbf{x}_B)$ and $\phi_\pm(\mathbf{x}_B, \mathbf{x}_A)$ into the probability equation:

$$\mathrm{d}P = \mathrm{d}^3\mathbf{x}_A \mathrm{d}^3\mathbf{x}_B \left(\frac{1}{2} |\phi_A(\mathbf{x}_A)\phi_B(\mathbf{x}_B)|^2 + \frac{1}{2} |\phi_B(\mathbf{x}_B)\phi_A(\mathbf{x}_A)|^2 \right).$$

This expression simplifies to:

$$\mathrm{d}P = \mathrm{d}^3\mathbf{x}_A \mathrm{d}^3\mathbf{x}_B |\phi_A(\mathbf{x}_A)|^2 |\phi_B(\mathbf{x}_B)|^2.$$

Conclusion: This probability matches the one for distinguishable particles, indicating that in situations where wave functions do not overlap, identical particles can be treated as distinguishable. Thus, there is no need for symmetrization or antisymmetrization in such cases.

Occupation Numbers

21.8 Occupation Numbers

Consider a system of N identical particles, each living in a vector space V. The N-particle quantum system lives in $V^{\otimes N}$.

- If the particles are bosons, the states lie in $\mathrm{Sym}^N V$.
- If the particles are fermions, the states lie in $\mathrm{Anti}^N V$.
- To construct symmetric and antisymmetric states, we can start with a basis state in $V^{\otimes N}$ and apply the symmetrizer $\hat{S}$ or the antisymmetrizer $\hat{A}$.
- Many different basis states in $V^{\otimes N}$ can lead to the same state in $\mathrm{Sym}^N V$ or $\mathrm{Anti}^N V$ after applying the projectors.
- The states in $\mathrm{Sym}^N V$ and $\mathrm{Anti}^N V$ are thus long lists of superpositions of basis states.

Why Use Occupation Numbers:

- Occupation numbers provide an economical yet complete way to specify states in $\mathrm{Sym}^N V$ and $\mathrm{Anti}^N V$.
- They help distinguish basis states in $V^{\otimes N}$ that, after symmetrization or antisymmetrization, are linearly independent.
- Occupation numbers give a simplified description of the physical states.

21.8 Occupation Numbers (continued)

Formal Definition:

- Let $\{|u_i\rangle\}$ with $i = 1, 2, \ldots$ form an orthonormal basis of V: $V = \text{span}\{|u_1\rangle, |u_2\rangle, \ldots\}$.
- For the N-particle system, basis states in $V^{\otimes N}$ take the form:

$$|u_{i_1}\rangle_{(1)} \otimes |u_{i_2}\rangle_{(2)} \otimes \cdots \otimes |u_{i_N}\rangle_{(N)},$$

where $i_1, \ldots, i_N$ can take any values in the list of basis state labels of V.

Assigning Occupation Numbers:

- For any basis state $|\omega\rangle \in V^{\otimes N}$, we assign a set of occupation numbers $\{n_1, n_2, \ldots\}$ where $n_i \geq 0$.
- The integer n_i denotes the number of times $|u_i\rangle$ appears in the basis state $|\omega\rangle$:

$$\underbrace{|u_1\rangle}_{n_1}, \underbrace{|u_2\rangle}_{n_2}, \cdots, \underbrace{|u_i\rangle}_{n_i}, \cdots$$

- By inspecting $|\omega\rangle$, all occupation numbers $n_1, n_2, \ldots$ can be identified.
- The list of the nonzero occupation numbers is finite and has at most N elements because we are describing states of N particles.

Example (two particles in a three-state system)

Occupation numbers for two particles in a three-state system.

To illustrate the use of occupation numbers, consider:

- A single-particle state space V with three basis states $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$.
- Two particles, so that $N = 2$.
- There are nine basis states for $V \otimes V = V^{\otimes 2}$.

For any basis state, we define three occupation numbers forming a list $\{n_1, n_2, n_3\}$, with $n_i \in \{0, 1, 2\}$, indicating how many particles are in $|u_i\rangle$.

Same States for both particles:

- If both particles are in the same state, we have the following basis vectors:

$$|u_1\rangle_{(1)} \otimes |u_1\rangle_{(2)} \Rightarrow \{2, 0, 0\}, \quad |u_2\rangle_{(1)} \otimes |u_2\rangle_{(2)} \Rightarrow \{0, 2, 0\}, \quad |u_3\rangle_{(1)} \otimes |u_3\rangle_{(2)} \Rightarrow \{0, 0, 2\}.$$

- These basis vectors are symmetric under the transposition (12), hence suitable for representing bosons.

Example (continued)

Different States for Each Particle:

- If the two particles are in different states, the occupation numbers are:

$$|u_1\rangle_{(1)} \otimes |u_2\rangle_{(2)}, |u_2\rangle_{(1)} \otimes |u_1\rangle_{(2)} \Rightarrow \{1, 1, 0\},$$

$$|u_1\rangle_{(1)} \otimes |u_3\rangle_{(2)}, |u_3\rangle_{(1)} \otimes |u_1\rangle_{(2)} \Rightarrow \{1, 0, 1\},$$

$$|u_2\rangle_{(1)} \otimes |u_3\rangle_{(2)}, |u_3\rangle_{(1)} \otimes |u_2\rangle_{(2)} \Rightarrow \{0, 1, 1\}.$$

- Each line shows two $V \otimes V$ basis states with the same occupation numbers due to exchange degeneracy.
- For bosons: Pick symmetric superpositions on each line to get three states in $\mathrm{Sym}^2 V$.
- For fermions: Pick antisymmetric superpositions on each line to get three states in $\mathrm{Anti}^2 V$.
- While there are a total of nine basis states in $V \otimes V$, there are only six possible sets of occupation numbers.
- All six can be used to describe states of bosons. Only three can be used to describe states of fermions

Example (continued)

Pictorial Representation:

- For basis states $|u_1\rangle$, $|u_2\rangle$, and $|u_3\rangle$, we use three lines:
  - The lowest line for $|u_1\rangle$,
  - The middle line for $|u_2\rangle$,
  - The top line for $|u_3\rangle$.

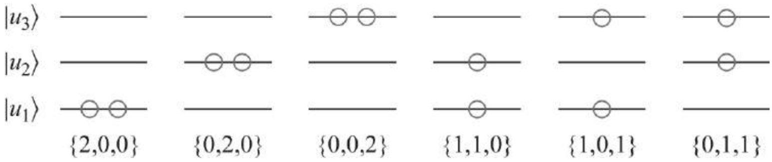


Figure: Two particles in a three-state system. The three levels are indicated by horizontal lines, and the particles are shown as small circles. There are six possible sets of occupation numbers.

- Each state is represented by a small circle at the corresponding level. Figure illustrates six configurations for possible occupation numbers:
  - All six configurations represent states for bosons.
  - Only the last three configurations represent states for fermions.

Occupation Numbers and Basis States

General Situation:

- Each basis state $|\omega\rangle \in V^{\otimes N}$ and the state obtained by acting with a permutation operator have exactly the same occupation numbers.
- Two basis states in $V^{\otimes N}$ with identical occupation numbers can be mapped into each other by a permutation, leading to the same state in $\mathrm{Sym}^N V$ or in $\mathrm{Anti}^N V$ (up to a sign).
- States with different occupation numbers cannot be permuted into each other, hence defining unique states in $\mathrm{Sym}^N V$ and $\mathrm{Anti}^N V$.

Defining Symmetric States

- Given a basis state in $V^{\otimes N}$, the associated symmetric basis state can be labeled by occupation numbers $\{n_1, n_2, \ldots\}$.
- Occupation numbers must add up to N:

$$|n_1, n_2, \ldots\rangle_s \in \mathrm{Sym}^N V, \quad n_i \geq 0, \quad \sum_i n_i = N.$$

- The symmetric state is constructed as:

$$|n_1, n_2, \ldots\rangle_S \equiv c_S \hat{S}(\underbrace{|u_1\rangle \ldots |u_1\rangle}_{n_1 \text{ times}} \otimes \underbrace{|u_2\rangle \ldots |u_2\rangle}_{n_2 \text{ times}} \otimes \cdots),$$

  where c_S is a normalization constant.
- More briefly, the symmetric state is given by:

$$|n_1, n_2, \ldots\rangle_S \equiv c_S \hat{S} \left(|u_1\rangle^{\otimes n_1} \otimes |u_2|^{\otimes n_2} \otimes \cdots\right)$$

Defining Symmetric States (continued)

Orthogonality of States Defined by Occupation Numbers:

- States in $\mathrm{Sym}^N V$ defined by different occupation numbers are orthogonal:

$$_S\langle n_1', n_2', \ldots | n_1, n_2, \ldots \rangle_S = \delta_{n_1, n_1'} \delta_{n_2, n_2'} \cdots .$$

- The space $\mathrm{Sym}^N V$ is spanned by all possible occupation numbers:

$$\mathrm{Sym}^N V = \mathrm{Span} \left\{ |n_1, n_2, \ldots \rangle_S \,\middle|\, \sum_i n_i = N, n_i \geq 0, \forall i \right\} .$$

Defining Antisymmetric States

Construction of Antisymmetric Spaces:

- For fermions, no occupation number can exceed one due to Pauli's exclusion principle.
- Orthonormal basis states in $\mathrm{Anti}^N V$ are defined as:

$$|n_1, n_2, \ldots\rangle_A \equiv c_A \hat{A} \left(|u_1\rangle^{\otimes n_1} \otimes |u_2\rangle^{\otimes n_2} \otimes \cdots \right), \quad n_i \in \{0, 1\}, \quad \sum_i n_i = N,$$

  Any state with an occupation number two or larger is killed by $\hat{A}$, because it cannot be antisymmetrized.
- These states indeed satisfy

$$_A\left\langle n_1', n_2', \ldots \,\middle|\, n_1, n_2, \ldots \right\rangle_A = \delta_{n_1, n_1'} \delta_{n_2, n_2'} \cdots$$

- The space relevant to identical fermions is

$$\mathrm{Anti}^N\, V = \mathrm{Span} \left\{ |n_1, n_2, \ldots\rangle_A \,\middle|\, \sum_i n_i = N, n_i \in \{0, 1\}, \forall i \right\}.$$

Axioms of Quantum Mechanics

- Let us now state the axioms, denoted as A1, A2, A3, and A4, valid for any isolated quantum system
- The axiomatic formulation described below follows the **Copenhagen interpretation** of quantum mechanics developed by Bohr, Heisenberg, and others. Perhaps one day it will be improved and replaced with a less mysterious one, but to date, this formulation is consistent with all known facts about quantum mechanics.

A1. States of the system

The complete description of a quantum system is given by a ray in a Hilbert space $\mathcal{H}$.

Remarks:

- A *ray* in a vector space is a nonzero vector $|\Psi\rangle$ with the equivalence relation $|\Psi\rangle \simeq c|\Psi\rangle$ for any nonzero $c \in \mathbb{C}$. Any vector in this ray is a representative of the state of the system. The vector $|\Psi\rangle$ is also called a *wave function*.
- Affirming that the state gives a complete description of the system, the axiom implies that the state describes the *most* that can be known about the system.
- The state $|\Psi\rangle$ of the system has a representative with unit norm. This representative is a normalized state or wave function.
- However complicated the quantum system and however many particles it contains, just *one state*, one wave function, represents the full quantum state of the system.

A2. Observables

Hermitian operators on the state space $\mathcal{H}$ are observables.

Remarks:

- An observable of a system is a property of the system that can be measured. This postulate says that such properties arise from Hermitian operators.
- The spectral theorem (section 15.6) implies that any observable $\hat{A} = \hat{A}^\dagger$ can be written as the sum:

$$\hat{A} = \sum_k a_k P_k \tag{29}$$

  where the sum runs over all the *different* eigenvalues a_k of $\hat{A}$, and the P_k are a complete set of orthogonal projectors into the corresponding eigenspaces.
- The projectors P_k satisfy:

$$P_k^\dagger = P_k, \quad P_k P_l = \delta_{kl} P_k, \quad \sum_k P_k = 1$$

- If an eigenvalue is nondegenerate, the associated projector is rank one. If an eigenvalue has a multiplicity $l > 1$, the associated projector is rank l and projects into an l-dimensional eigenspace.

A3. Measurement

Let P_k, with $k = 1, \ldots$, denote a complete set of orthogonal projectors, and let $\mathcal{H}_k$ denote the subspace P_k projects into. Measurement along this set of projectors is a process in which the state Ψ is projected to $\mathcal{H}_k$ with probability $p(k)$ given by

$$p(k) = \langle \Psi | P_k | \Psi \rangle = \| P_k | \Psi \rangle \|^2 .$$

The normalized state after measurement is

$$\frac{P_k | \Psi \rangle}{\| P_k | \Psi \rangle \|} .$$

Measuring an observable $\hat{A}$ is measuring along the complete set of orthogonal projectors associated with its spectral decomposition (29). The probability $p(k)$ for the state to be projected to $\mathcal{H}_k$ is the probability of measuring a_k.

A3. Measurement (Remaks)

- Measurement of an observable is a nondeterministic physical process. We cannot in general predict the result of the measurement, just the probabilities for the various possible results.
- The measurement axiom does not give any prescription for measuring the state Ψ itself. The state is the full description of the system, but it cannot be directly measured. We can only measure observables, and such measurements give us some information about the state.
- The probabilities $p(k)$ add up to one, as they should. Using the completeness of the set of projectors, we find

$$\sum_k p(k) = \sum_k \langle \Psi | P_k | \Psi \rangle = \langle \Psi | \left(\sum_k P_k \right) | \Psi \rangle = \langle \Psi | \Psi \rangle = 1.$$

A3. Measurement (Remaks)

- When measuring $\hat{A}$, if the eigenvalue a_k is nondegenerate with eigenvector $|k\rangle$, then $P_k = |k\rangle\langle k|$, and the probability $p(k)$ is

$$p(k) = \langle\Psi|k\rangle\langle k|\Psi\rangle = |\langle k|\Psi\rangle|^2.$$

- If the eigenvalue a_k is degenerate with multiplicity l, the associated eigenspace is spanned by l orthonormal eigenvectors $|k;1\rangle, \ldots, |k;l\rangle$, and

$$P_k = \sum_{i=1}^{l} |k;i\rangle\langle k;i|,$$

$$p(k) = \left(\sum_{i=1}^{l}\langle\Psi|k;i\rangle\langle k;i|\Psi\rangle\right) = \sum_{i=1}^{l} |\langle k;i|\Psi\rangle|^2.$$

A3. Measurement (Remaks)

- Measurement along an orthonormal basis $\{|i\rangle\}$ means measuring along the complete set of rank-one orthogonal projectors $P_i = |i\rangle\langle i|$.
- The probability $p(i)$ of being found in the state $|i\rangle$ arising by projection via P_i is

$$p(i) = \langle \Psi | i \rangle \langle i | \Psi \rangle = |\langle i | \Psi \rangle|^2.$$

- When we say we are measuring an orthogonal projector P, we mean treating P as a Hermitian operator, with eigenvalues one and zero.

A4. Dynamics

Time evolution is unitary: given any state $|\Psi, t_0\rangle$ of the system at time t_0, the state $|\Psi, t_1\rangle$ at time t_1 is obtained by the action of a unitary operator $\mathcal{U}(t_1, t_0)$:

$$|\Psi, t_1\rangle = \mathcal{U}(t_1, t_0)|\Psi, t_0\rangle.$$

Remarks:

- Time evolution is deterministic: if the state is known exactly at some time, it is known exactly at a later time.
- The same operator $\mathcal{U}(t_1, t_0)$ evolves any possible state of the system at time t_0.
- Unitary time evolution means the state satisfies the Schrödinger equation $i\hbar\frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$ with $\hat{H}$ as the Hamiltonian, a Hermitian operator with units of energy.
- Thus, axiom A4 implies that any quantum system has a Schrödinger equation that controls the time evolution of the wave function.

Composite System and Symmetrization Postulate

In setting up certain quantum mechanical systems, two guiding principles are useful but don't directly follow from the four axioms:

- **P1. Composite System Postulate:**
  Assume system A has a state space $\mathcal{H}_A$, and system B has a state space $\mathcal{H}_B$. The state space of the composite system AB is the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$. If system A is in $|\Psi_A\rangle$ and system B is in $|\Psi_B\rangle$, then the state of AB is $|\Psi_A\rangle \otimes |\Psi_B\rangle$.
- **P2. Symmetrization Postulate:**
  For a system with N identical particles, the physically realized states are either totally symmetric (bosons) or totally antisymmetric (fermions) under particle exchange.

Additional Notes:

- The composite system postulate (P1) facilitates the implementation of Axioms A1 to A4 for composite systems.
- The symmetrization postulate (P2) resolves the problem of *exchange degeneracy*. In fact, this postulate is proven in relativistic quantum field theory under some weak set of assumptions.
- In quantum systems with two spatial dimensions, particles that are neither bosons nor fermions can exist.