1. Classical Field Theory.

1.1 Lagrange formalism for particles

Classical mechanics

$L(q_n, \dot{q}_n)$ Lagrange function [assume no explicit time dependence]

q_n $n = 1, \cdots, N$ generalized cooridinates

$S[q_n] = \int_{t_1}^{t_2} L(q_n(t), \dot{q}_n(t))\, dt$ Action

Functional of orbits $q_n(t)$ with fixed starting and end points $q_n(t_1)$, $q_n(t_2)$

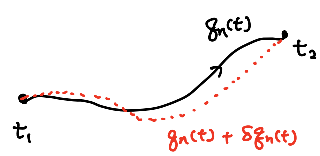
Equation of motion follows from the "action principle". The orbit taken by the physical system is the one where the action is stationary:

$$\delta S[q_n] = 0$$

This implies the Euler-Lagrange equations

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = 0}$$

Reminder :



$$\delta S[q_n] = S[q_n + \delta q_n] - S[q_n]$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_n}\delta q_n + \frac{\partial L}{\partial \dot{q}_n}\delta \dot{q}_n \right)$$

$$\delta \dot{q}_n \equiv \frac{d}{dt}\delta q_n$$

$$= \underbrace{\left[\frac{\partial L}{\partial \dot{q}_n}\delta q_n \right]_{t_1}^{t_2}}_{=0} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} \right)\delta q_n \stackrel{!}{=} 0$$

since $\delta q_n(t_1) = \delta q_n(t_2) = 0$

For arbitrary δq_n this implies the E.L. equation

Transition to the Hamiltonian:

cannonically conjugate momenta $\quad p_n \equiv \dfrac{\partial L}{\partial \dot{q}_n} \qquad (*)$

Then $\quad H(q_n, p_n) \equiv \sum_n p_n \dot{q}_n - L(q_n, \dot{q}_n)$

$\dot{q}_n = \dot{q}_n(q_n, p_n)$ by solving $(*)$ for $\dot{q}_n$

The canonical coordinates fulfill the equations

$$\{q_n, p_m\}_P = \delta_{nm} \qquad \{q_n, q_m\}_P = \{p_n, p_m\}_P = 0$$

$$\dot{q}_n = \{q_n, H\}_P = \frac{\partial H}{\partial p_n}$$

$$\dot{p}_n = \{p_n, H\}_P = -\frac{\partial H}{\partial q_n}$$

Poisson bracket

$$\{A, B\}_P = \frac{\partial A}{\partial q_n}\frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n}\frac{\partial B}{\partial q_n}$$

Canonical quantisation

$L(q_n, \dot{q}_n)$ given. Interpret $q_n, \dot{q}_n, p_n$ as operators on a Hilbert space.

Impose the commutation relations $[\hbar \equiv 1]$

$$[q_n, p_m] = i\delta_{nm}$$

$$[q_n, q_m] = [p_n, p_m] = 0$$

(at a fixed time)

The time evolution of the operators is

$$\dot{q}_n = \frac{1}{i}[q_n, H]$$

Note: simply $\{\ ,\ \}_P \to \frac{1}{i}[\ ,\ ]$

1.2 Lagrange - formalism for fields

- A field is a quantity defined at every point of space and time $(\vec{x}, t)$

$$\phi(\vec{x}, t)$$

NOTE: the concept of position has been relegated from a dynamical variable in classical mechanism to a mere label in field theory

- An example: E-M field

$$\left.\begin{array}{l}\text{Electric field: } \vec{E}(\vec{x}, t) \\ \text{Magnetic field: } \vec{B}(\vec{x}, t)\end{array}\right\}: \text{ spatial 3-vectors}$$

derive these two 3-vectors from a single 4-component field

$$\underbrace{A^{\mu}}_{\text{a vector in spacetime}}(\vec{x}, t) = (\phi, \vec{A}) \qquad [\mu = 0, 1, 2, 3]$$

The electric and magnetic fields are

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \qquad \text{and} \qquad \vec{B} = \nabla \times \vec{A}$$

The dynamics of the field is governed by a Lagrangian

In relativistic QFT one assumes that L is an integral over Lagrangian density

$$L = \int d^3\vec{x}\ \mathcal{L}(\phi_n(x),\ \vec{\nabla}\phi_n(x),\ \dot{\phi}_n(x))$$

$$= \int d^3\vec{x}\ \underset{\substack{\Uparrow \\ \text{all at one space-point : local}}}{\mathcal{L}}(\partial_\mu \phi_n(x),\ \phi_n(x))$$

It is difficult to construct non-local Lagrangians, which are compatible with relativistic causality.

E.-L. equation:

$$\delta S = \delta \int \underbrace{dt\, d^3\vec{x}}_{d^4x}\ \mathcal{L}(\phi_n,\ \partial_\mu \phi_n)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_n}\delta\phi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)}\delta(\partial_\mu \phi_n) \right]$$

$$\left.\begin{array}{l}\text{Integration by parts and} \\ \text{assume } \phi_n(x) \to 0 \text{ for } |\vec{x}| \to \infty\end{array}\right\} \Rightarrow \quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_n} = 0$$

Canonically conjugated fields : $\quad \pi_n(x) \equiv \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \phi_n(x))}$

Hamilton density:

$$H = \sum_n \int d^3\vec{x}\ \pi_n(x)\, \partial_0 \phi_n(x) - \int d^3\vec{x}\ \mathcal{L} \equiv \int d^3x\ \mathcal{H}(x)$$

$$\Rightarrow \mathcal{H}(x) = \sum_n \pi_n \partial_0 \phi_n - \mathcal{L}$$

Suppose space were discrete with points $\vec{x}$ on a lattice, then there is a set $\phi_{n,\vec{x}}(t)$ of cannonical coordinates, and the previous discussion applies with substitutions

$$q_n(t) \rightarrow \phi_{n,\vec{x}}(t) \longrightarrow \phi_n(t,\vec{x}) \equiv \phi(x)$$

$$\sum_n \rightarrow \sum_{n,\vec{x}} \longrightarrow \sum_n \int d^3\vec{x}$$

$$\delta_{nm} \longrightarrow \delta_{nm}\,\delta_{\vec{x}\vec{y}} \longrightarrow \delta_{nm}\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$L(q_n(t), \dot{q}_n(t)) \longrightarrow L[\phi_n(t), \dot{\phi}_n(t)] \rightarrow \mathcal{L}(\phi_n(x), \partial_\mu\phi_n(x))$$

Cannonical quantization

Impose the equal-time commutation relations

$$[\phi_n(t,\vec{x}), \pi_m(t,\vec{y})] = i\,\delta_{nm}\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$[\phi_n(t,\vec{x}), \phi_m(t,\vec{x})] = [\pi_n(t,\vec{x}), \pi_m(t,\vec{y})] = 0$$

The E.L. eqution is equivalent to the Heisenberg equation

$$\dot{\phi}_n(t,\vec{x}) = \frac{1}{i}[\phi_n(t,\vec{x}), H]$$

for the field operator. The formal solution is

$$\phi_n(t,\vec{x}) = e^{iHt}\,\phi_n(0,\vec{x})\,e^{-iHt}$$

[Since H does not explicitly depend on t]

NOTE : • If the commutation relations are imposed at one time, they are preserved at all times

$$[\phi_n(t,\vec{x}), \pi_m(t,\vec{x})] = [e^{iH(t-t_0)}\phi_n(t_0,\vec{x})e^{-iH(t-t_0)}, e^{iH(t-t_0)}\pi_m(t_0,\vec{x})e^{-iH(t-t_0)}]$$

$$= e^{iH(t-t_0)}\underbrace{[\phi_n(t_0,\vec{x}), \pi_m(t_0,\vec{x})]}_{=i\delta_{nm}\delta^{(3)}(\vec{x}-\vec{y})}e^{-iH(t-t_0)}$$

$$= i\,\delta_{nm}\,\delta^{(3)}(\vec{x}-\vec{y})$$

• Unless otherwise mentioned one uses the Heisenberg picture in QFT, where the time-dependence is in the operators, not the states.

An Example : The Klein – Gordon Equation

consider L for a real scalar field $\phi(\vec{x}, t)$

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} m^2 \phi^2$$

$$= \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\vec{\nabla}\phi)^2 - \frac{1}{2} m^2 \phi^2$$

use the Minkowski space metric $g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

we identify the kinetic energy of the field as

$$T = \int d^3x \, \frac{1}{2} \dot{\phi}^2$$

the potential energy as

$$V = \int d^3x \left[\underbrace{\frac{1}{2} (\vec{\nabla}\phi)^2}_{\text{gradient energy}} + \underbrace{\frac{1}{2} m^2 \phi^2}_{\text{"true" potential energy}} \right]$$

To determine the equation of motion (EOM) for ϕ, we compute

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \equiv (\dot{\phi}, -\nabla \phi)$$

The E.-L. equation is then

$$\ddot{\phi} - \nabla^2 \phi + m^2 \phi = 0 \qquad \text{or} \qquad \partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

Klein - Gorden Equation

If we consider the Lagrangian with general potential $V(\phi)$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi) \Rightarrow \partial_\mu \partial^\mu \phi + \frac{\partial V}{\partial \phi} = 0$$

Klein-Gorden Equation:

Schrödinger eqn : $$i\hbar \frac{\partial}{\partial t}\psi(\vec{x},t) = \underset{\substack{\uparrow \\ \text{Hamiltonian}}}{-\frac{\hbar^2}{2m}} \nabla^2 \psi(\vec{x},t) \qquad H = \frac{\hat{p}^2}{2m}$$

For the relativistic motion:

$$H = \sqrt{p^2c^2 + m^2c^4} = \underset{\substack{\uparrow \\ \text{rest energy}}}{mc^2} + \underset{\substack{\uparrow \\ \text{nonrelativistic Hamiltonian}}}{\frac{p^2}{2m}} + \cdots$$

Then the Schrödinger eqn can be written as

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{x},t) = \sqrt{-\hbar^2c^2\nabla^2 + m^2c^4}\,\psi(\vec{x},t)$$

$\downarrow$ square the differential operators on each side

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi(\vec{x},t) = (-\hbar^2c^2\nabla^2 + m^2c^4)\psi(\vec{x},t)$$

$\downarrow$ $\hbar = c = 1$

$$\ddot{\psi} - \nabla^2\psi + m^2\psi = 0 \qquad \text{or} \qquad \partial_\mu \partial^\mu \psi + m^2\psi = 0$$

K-G wave function $\psi(t,\vec{x})$ is a Lorentz-invariant function.

Since $|\psi|^2$ is also invariant, this cannot represent a probability density.

A density transforms as time-like component of a 4-vector, due to Lorentz contraction of volume element.

One can define

$$\rho = \frac{i}{2m}(\psi^*\dot{\psi} - \dot{\psi}^*\psi)$$

$$\vec{j} = \frac{1}{2im}(\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi)^*\psi)$$

follow a continuity equation : $\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0 \Leftrightarrow \partial_\mu j^\mu = 0$

However, ρ is not necessarily positive (unlike $|\psi|^2$). Therefore, it may well be considered as the density of a conserved quantity (the electric charge, e.g.), but not as a positive probability.

The 2nd problem : negative energy solution

plane wave solution : $\psi(t, \vec{x}) = N e^{-i(Et - \vec{p}\cdot\vec{x})}$

$\hookrightarrow \quad E^2 = |\vec{p}|^2 + m^2 \quad \Rightarrow \quad E = \pm\sqrt{|\vec{p}|^2 + m^2}$

The spectrum is no longer bounded from below.

E.g. Maxwell's Equations

From the Lagrangian $\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2$

NOTE: $\mathcal{L}$ has no kinetic term $\dot{A}_0^2$ for A_0

we compute

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)}\left[-\frac{1}{2}\partial_\alpha A_\beta \partial_\rho A_\sigma g^{\rho\alpha} g^{\sigma\beta} + \frac{1}{2}\partial_\alpha A_\beta \partial_\rho A_\sigma g^{\beta\alpha} g^{\sigma\rho}\right]$$

$$\boxed{\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\nu)} = \delta^\mu_\alpha \delta^\nu_\beta} \cdots\cdots\to = -\frac{1}{2}\delta^\mu_\alpha \delta^\nu_\beta \partial_\rho A_\sigma g^{\rho\alpha} g^{\sigma\beta} - \frac{1}{2}\partial_\alpha A_\beta \delta^\mu_\rho \delta^\nu_\sigma g^{\rho\alpha} g^{\sigma\beta}$$

$$+ \frac{1}{2}\delta^\mu_\alpha \delta^\nu_\beta \partial_\rho A_\sigma g^{\beta\alpha} g^{\sigma\rho} + \frac{1}{2}\partial_\alpha A_\beta \delta^\mu_\rho \delta^\nu_\sigma g^{\beta\alpha} g^{\sigma\rho}$$

$$= -\frac{1}{2}\partial^\mu A^\nu - \frac{1}{2}\partial^\mu A^\nu + \frac{1}{2}\partial_\rho A^\rho g^{\mu\nu} + \frac{1}{2}\partial_\alpha A^\alpha g^{\mu\nu}$$

$$= -\partial^\mu A^\nu + \partial_\rho A^\rho g^{\mu\nu}$$

Then we have

$$\partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right] = -\partial^2 A^\nu + \partial^\nu[\partial_\rho A^\rho]$$

$$= -\partial_\mu[\partial^\mu A^\nu - \partial^\nu A^\mu] \equiv -\partial_\mu F^{\mu\nu}$$

where the field strength is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

Using $F^{\mu\nu}$, the Maxwell Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

1.3 Noether's theorem for fields

- The Noether theorem establishes a relation between symmetries and conserved quanties
- There is no difference between classical and quantum field theory ; $\varphi(x)$ can be a classical field or quantum field operator.

$$S[\varphi] = \int d^4x\, \mathcal{L}(\varphi, \partial_\mu\varphi)$$

A symmetry is a transformation of the fields, which leaves action invariant

$$\varphi'(x) = \varphi(x) + \varepsilon F(x) \longleftarrow \begin{array}{l} \varepsilon : \text{ a constant ; global symmetry} \\ F : \text{ depends } \varphi(x) \text{, and } \partial_\mu\varphi(x) \end{array}$$

$$0 = \delta S = \varepsilon \int d^4x \frac{\delta S[\varphi]}{\delta\varphi(x)} F(x)$$

before the e.o.m. are staisfied

Example : translation

$$\varphi'(x) = \varphi(x+a) = \varphi(x) + a^\mu\partial_\mu\varphi(x) + \cdots \Rightarrow \begin{cases} a^\mu \sim \varepsilon \\ \partial_\mu\varphi \sim F \end{cases}$$

Noether theorem : For every continuous symmetry there is a conserved current

$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi(x))} F(x) - K^\mu(x) \qquad [\ \partial_\mu j^\mu(x) = 0\ ]$$

and charge

$$Q \equiv \int d^3\vec{x}\, j^0(x) \qquad [\ \frac{dQ}{dt} = 0\ ]$$

Derivation :

Since the action $S[\varphi(x)]$ is invariant by assumption, $\mathcal{L}$ can only change by a total derivative . i.e.

$$\delta S = \varepsilon \int d^4x\, \partial_\mu K^\mu(x) \qquad (1)$$

for some $K^\mu(x)$ (In particular $K^\mu \equiv 0$ if $\mathcal{L}$ is invariant , not only $S[\varphi]$)

On the other hand

$$\delta S = \int d^4x \, \frac{\partial \mathcal{L}}{\partial \varphi} \underbrace{\delta \varphi}_{\varepsilon F(x)} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \underbrace{\delta (\partial_\mu \varphi)}_{\varepsilon \partial_\mu F(x)}$$

use E.L. equation to eliminate $\partial \mathcal{L}/\partial \varphi$, then we have

$$\delta S = \int d^4x \; \varepsilon F(x) \; \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right] + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \, \varepsilon \, \partial_\mu F(x)$$

$$= \int d^4x \; \varepsilon \; \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} F \right] \qquad (2)$$

Taking (2) minus (1) gives $\partial_\mu j^\mu(x) = 0$ for $j^\mu(x)$ as defined above.

Also

$$\frac{dQ}{dt} = \int d^3\vec{x} \; \partial^0 j^0(x) \underset{\partial^\mu j_\mu = 0}{=} - \int d^3\vec{x} \; \vec{\nabla} \cdot \vec{j}(x) = 0 .$$

assuming as usual that the fields vanish as $|\vec{x}| \to \infty$

NOTE : The conservation of the current is only true for the fields satisfying the field equations, because the E.L. equation was used to obtain (2). In particular, in the quantum theory, this will not be true of every field configuration we sum over in the path integral, only the saddle-point configurations.

An example: Translation and the Energy-Momentum Tensor.

Consider the infinitesimal translation

$$x^\nu \to x^\nu - \varepsilon^\nu \quad \Rightarrow \quad \phi_n(x) \to \phi_n(x) + \varepsilon^\nu \partial_\nu \phi_n(x)$$

$$\mathcal{L}(x) \to \mathcal{L}(x) + \varepsilon^\nu \partial_\nu \mathcal{L}(x) \longleftarrow \varepsilon^\nu \partial_\mu [\delta^\mu_\nu \mathcal{L}(x)]$$

Then four conserved currents $(j^\mu)_\nu$, one for each of ε^ν, $\nu = 0, 1, 2, 3$

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\nu \phi_n - \delta^\mu_\nu \mathcal{L}(x) \equiv T^\mu{}_\nu$$

Energy-Momentum tensor

It satisfies $\partial_\mu T^\mu{}_\nu = 0$

Four conserved quantities are given by

$$E = \int d^3x \; T^{00} \quad , \quad P^i = \int d^3x \; T^{0i}$$

the total energy of the field configuration

the total momentum

An example: the scalar field theory : $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$

$$T^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$$

Using E.O.M. for ϕ, $(\partial^2 + m^2)\phi = 0$. $\Rightarrow$ $\partial_\mu T^{\mu\nu} = 0$

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu(\partial^\mu \phi \, \partial^\nu \phi) - \partial^\nu \mathcal{L} \\
&= \partial^2 \phi \, \partial^\nu \phi + \partial^\mu \phi \, \partial_\mu(\partial^\nu \phi) - \partial^\nu \left(\tfrac{1}{2} \partial_\mu \phi \partial^\mu \phi - \tfrac{1}{2} m^2 \phi^2 \right) \\
&= -m^2 \phi \, \partial^\nu \phi + \partial^\mu \phi \, \partial_\mu(\partial^\nu \phi) - \tfrac{1}{2} \partial^\nu(\partial_\mu \phi) \partial^\mu \phi - \tfrac{1}{2} \partial_\mu \phi \, \partial^\nu(\partial^\mu \phi) \\
&\qquad + m^2 \phi \, \partial^\nu \phi \\
&= 0
\end{aligned}$$

the conserved energy $\quad E = \int d^3x \; \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2$

" " momentum $\quad P^i = \int d^3x \; \dot{\phi} \, \partial^i \phi$

1.4 Lorentz transformation

Lorentz invariant is symmetry under rotations and boosts.

$\boxed{\text{The rotation groups } SO(n)}$

E.g. Two-dimensional rotation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The line element dl^2 in the phyical space $\mathbb{R}^3$, measuring the distance between any two points $\vec{x}$ and $\vec{x} + d\vec{x}$

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx^2 + dy^2 + dz^2$$

which is invariant under linear transformations

$$\vec{x} \longrightarrow \vec{x}' = R\,\vec{x}$$

R : orthogonal matrices $\quad R\,R^T = R^T \cdot R = 1_3$

Thus, $\det R = 1$: proper orthogonal transformations

rotations in the physical space.

$\det R = -1$: improper orthogonal transformation

such as space reflection

The set of all proper orthogonal transformations form the group of rotations denoted by $SO(3)$

S : special, $\det R = 1$

The set of all orthogonal transformations form group $O(3)$.

$SO(3)$ is a subgroup of $O(3)$

$\boxed{\text{The basic definition of group:}}$

A set $G = \{a, b, c \cdots\}$ is called a group under the operation of "multipication" denoted $\cdot$ if the following axioms hold:

(i) G is closed under the operation "$\cdot$" (封闭性) Closure

If $a, b \in G$, then $a \cdot b \in G$

(ii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in G$ (结合律) Associativity

(iii) $\exists\ e \in G$ such that $a \cdot e = e \cdot a = a$ for $\forall\ a \in G$ (单位元) Identity

(iv) For $\forall\ a \in G$, $\exists\ a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ (逆元) Invertibility

Isomorphism: Two groups G and G' are said isomorphic if there exists a
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one-to-one correspodence between their elements which preserves the law of group multiplication.

If $g_i \in G \leftrightarrow g_i' \in G'$ and $g_1 g_2 = g_3$ in G, then $g_1' g_2' = g_3'$ in G' and vice versa.

Homomorphism: A homomorphism from a group G to another group G' is a mapping
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(not necessarily one-to-one) which preserves group multiplication.

If $g_i \in G \rightarrow g_i' \in G'$ and $g_1 g_2 = g_3$ then $g_1' g_2' = g_3'$

Clearly, isomorphism is a special case of homomorphism. (双射)

Representations of a Group:

If there is a homomorphism from a group G to a group of operators $U(G)$ on a linear space V, we say that $U(G)$ forms a representation of the group G.

The dimension of the representation is the dimension of the vector space V.

A representation is said to be faithful if the homomorphism is also an isomorphism (i.e. one-to-one)

The representiation is a mapping $\quad g \in G \xrightarrow{U} U(g)$

where $U(g)$ is an operator on V, such that $\quad U(g_1)\, U(g_2) = U(g_3)$

U satisfy the same rules of multiplication

Consider the case of a finite-dimensional representation. Choose a set of basis vectors $\{\hat{e}_i,\ i=1,\cdots,n\}$ on V. The operators are then realized as $n\times n$ matrices $D(g)$ as follows:

$$U(g)\,|e_i\rangle = |e_j\rangle\, D(g)^j{}_i\,, \qquad g\in G$$

$$U(g_1)\,U(g_2)\,|e_i\rangle = U(g_1)\,|e_j\rangle\, D(g_2)^j{}_i = |e_k\rangle\, D(g_1)^k{}_j\, D(g_2)^j{}_i$$

$$= U(g_1 g_2)\,|e_i\rangle = |e_k\rangle\, D(g_1 g_2)^k{}_i$$

Since $\{\hat{e}_i\}$ from a basis, $\quad D(g_1)\, D(g_2) = D(g_1 g_2)$

the matrix multiplication is implied.

The group of matrices $D(G)$ forms a matrix representation of G.

$\boxed{\text{connected Lie group}}$ Important in physics

Groups of transformation $T(\theta)$ that are described by a finite set of real continuous parameters, say θ^a, with each element of the group connected to the identity by a path within the group.

Multiplication law : $T(\bar{\theta})\, T(\theta) = T(f(\bar{\theta}, \theta))$

$f^a(\bar{\theta}, \theta)$: a function of the $\bar{\theta}$s and θs.

Taking $\theta^a = 0$ as the coordinates of the identity, then

$$f^a(\theta, 0) = f^a(0, \theta) = \theta^a \qquad (*)$$

$$T(0)\, T(\theta) = T(\theta) = T(f(0, \theta))$$

The transformations of $T(\theta)$ must be represented on the Hilbert space by unitary operators $U(T(\theta))$. ← Identity operator $U = 1$ is unitary & linear.

At the finite neighborhood of the identity

$$U(T(\theta)) = 1 + i\theta^a t_a + \frac{1}{2}\theta^b\theta^c t_{bc} + \cdots$$

operators, t_a Hermitian, $t_{bc} = t_{cb}$

Suppose that the $U(T(\theta))$ form an ordinary representation.

$$U(T(\bar{\theta}))\, U(T(\theta)) = U(T(f(\bar{\theta}, \theta)))$$

Expanding in power of θ^a and $\bar{\theta}^a$

$$f^a(\theta, \bar{\theta}) = \theta^a + \bar{\theta}^a + f^a{}_{bc}\,\bar{\theta}^b\theta^c + \cdots$$

real coeficients.

No θ^2 or $\bar{\theta}^2$ since $(*)$

Then

$$\text{LHS of } (**) = [\ 1 + i\bar\theta^a t_a + \tfrac{1}{2}\bar\theta^b\bar\theta^c t_{bc}\ ]\ [\ 1 + i\theta^a t_a + \tfrac{1}{2}\theta^b\theta^c t_{bc}\ ]$$

$$= 1 + i(\bar\theta^a + \theta^a) t_a - \bar\theta^a\theta^b t_a t_b + \tfrac{1}{2}\bar\theta^b\bar\theta^c t_{bc} + \tfrac{1}{2}\theta^b\theta^c t_{bc} + \cdots$$

$$\text{RHS of } (**) = 1 + i f^a(\bar\theta,\theta) t_a + \tfrac{1}{2} f^b(\bar\theta,\theta) f^c(\bar\theta,\theta) t_{bc}$$

$$= 1 + i(\theta^a + \bar\theta^a + f^a{}_{bc}\bar\theta^b\theta^c) t_a + \tfrac{1}{2}(\theta^b + \bar\theta^b + \cdots)(\theta^c + \bar\theta^c + \cdots) t_{bc}$$

$$= 1 + i(\theta^a + \bar\theta^a + f^a{}_{bc}\bar\theta^b\theta^c) t_a + \tfrac{1}{2}(\theta^b\theta^c + \theta^b\bar\theta^c + \bar\theta^b\theta^c + \bar\theta^b\bar\theta^c) t_{bc}$$

The terms of order $1, \theta, \bar\theta, \theta^2, \bar\theta^2$ automatically match, from $\bar\theta\theta$ terms we have

$$-t_a t_b = i f^c{}_{ab} t_c + t_{ab}$$

Since $t_{ab} = t_{ba}$, then

$$-t_a t_b - i f^c{}_{ab} t_c = -t_b t_a - i f^c{}_{ba} t_c$$

$$\hookrightarrow [t_a, t_b] = i(-f^c{}_{ab} + f^c{}_{ba}) t_c$$

$$\hookrightarrow \qquad \equiv i C^c{}_{ab} t_c \qquad \longleftarrow \text{Lie algebra}$$

Note : Suppose that $f^a(\theta,\bar\theta) = \theta^a + \bar\theta^a$ ⟵ E.g. translations in space time

Then $f^a{}_{bc}$ vanish, so $[t_a, t_b] = 0$ ⟵ Abelian

In this case, for any integer N,

$$\underbrace{U(T(\tfrac{\theta}{N}))\cdots U(T(\tfrac{\theta}{N}))}_{N} = U(T(\tfrac{\theta}{N} + \cdots + \tfrac{\theta}{N})) = U(T(\theta))$$

Let $N \to \infty$, keep the 1st-order term in $U(T(\theta))$

$$U(T(\theta)) = \lim_{N\to\infty}\left(1 + \frac{i}{N}\theta^a t_a\right)^N$$

$$= \exp(i t_a \theta^a)$$

Lorentz Invariance :

Among the most important symmetries of relativistic QFT are those which arise from the Lorentz transformations themselves.

The space time interval ds^2

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$g_{\mu\nu} = g^{\mu\nu} \qquad \longleftarrow \text{inverse of } g_{\mu\nu}$$

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = G^{-1}$$

Any coordinate transformation $x^\mu \longrightarrow x'^\mu$ that satisfies

$$g_{\mu\nu}\, dx'^\mu dx'^\nu = g_{\mu\nu}\, dx^\mu dx^\nu \quad \text{or} \quad g_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} = g_{\rho\sigma}$$

is linear

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

arbitrary constants

a constant matrix satisfying $\quad g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \qquad (*)$

These transformations form a group

See : $\quad x^\mu \xrightarrow{(\Lambda, a)} x'^\mu \xrightarrow{(\bar{\Lambda}, \bar{a})} x''^\mu$

then $\quad x''^\mu = \bar{\Lambda}^\mu{}_\rho x'^\rho + \bar{a}^\mu = \bar{\Lambda}^\mu{}_\rho (\Lambda^\rho{}_\nu x^\nu + a^\rho) + \bar{a}^\mu$

$$= (\bar{\Lambda}^\mu{}_\rho \Lambda^\rho{}_\nu)\, x^\nu + (\bar{\Lambda}^\mu{}_\rho a^\rho + \bar{a}^\mu)$$

If, Λ, $\bar{\Lambda}$ satisfy $(*)$, then so does $\bar{\Lambda}\Lambda$

The transformation $T(\Lambda, a)$ indued on physical states satisfy

$$T(\bar{\Lambda}, \bar{a})\, T(\Lambda, a) = T(\bar{\Lambda}\Lambda,\ \bar{\Lambda} a + \bar{a})$$

Taking the determinant of $(*)$ gives $\quad (\det \Lambda)^2 = 1$, so $\Lambda^\mu{}_\nu$ has an inverse

$$(\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho \equiv g_{\nu\mu} g^{\rho\sigma} \Lambda^\mu{}_\sigma$$

See :

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}g_{\mu\nu} = g_{\rho\sigma} \quad \leftarrow \quad g_{\mu\nu}\Lambda^{\nu}{}_{\sigma} = \Lambda_{\mu\sigma}$$

$$\hookrightarrow \quad (\Lambda^T)_{\rho}{}^{\mu} g_{\mu\nu}\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma} \quad \leftarrow \text{ define } \quad (\Lambda^T)_{\mu}{}^{\nu} = \Lambda^{\nu}{}_{\mu}$$

$$\hookrightarrow \quad \Lambda^T G \Lambda = G$$

$$g_{\mu\nu} = g_{\rho\sigma}\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu} = \Lambda_{\sigma\mu}\Lambda^{\sigma}{}_{\nu}$$

$\downarrow$ raise μ on the both side

$$\delta^{\mu}{}_{\nu} = \Lambda_{\sigma}{}^{\mu}\Lambda^{\sigma}{}_{\nu}$$

Compare this with the definition of the inverse tranrformation Λ^{-1}

$$\underbrace{\Lambda^{-1}\Lambda = I}_{4\times 4 \text{ indentity matrix}} \quad \text{or} \quad (\Lambda^{-1})^{\mu}{}_{\sigma}\Lambda^{\sigma}{}_{\nu} = \delta^{\mu}{}_{\nu}$$

$$\Rightarrow \quad (\Lambda^{-1})^{\mu}{}_{\sigma} = \Lambda_{\sigma}{}^{\mu} \qquad \Rightarrow \quad (\Lambda^{-1})^{0}{}_{i} = \Lambda_{i}{}^{0} = -\Lambda^{i}{}_{0} \qquad (i = 1,2,3)$$

The whole group of $T(\Lambda, a)$ is Poincaré group

If $a^{\mu} = 0$, $T(\Lambda, 0)$ know as homogeneous Lorentz group

E.g. A rotation by θ about the x^3- axis

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A boost by v along the x^1- axis

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[with $\gamma = \dfrac{1}{\sqrt{1-v^2}}$

Note : Poincaré group 不仅作用于时空坐标，也作用于物理态统的 Hilbert 空间.

不仅仅表现为 4×4 矩阵

L has four components, each of which is connected in the sense that any one point can be connected to any other, but no Lorentz transformation in one component can be connected to another in another component.

Note that $\det \Lambda$ and $\operatorname{sgn} \Lambda^0{}_0$ are both continous functions of $\Lambda^\mu{}_\nu$.

Furthermore, $\det \Lambda = \pm 1$ and $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$

See :

$$\Lambda^T G \Lambda = G$$

$$\hookrightarrow \det (\Lambda^T G T) = \det G = -1 \quad \leftarrow \det G = 1 \times (-1)^3$$

$$\hookrightarrow (\det \Lambda)^2 = 1 \quad \leftarrow\text{---} \quad \det \Lambda^T = \det \Lambda \ ;$$

$$\det (A B) = \det(A) \det(B)$$

$$g^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta}$$

$$\hookrightarrow g^{00} = \Lambda^0{}_\alpha \Lambda^0{}_\beta g^{\alpha\beta} = (\Lambda^0{}_0)^2 + (\Lambda^0{}_i)(\Lambda^0{}_j) g^{ij}$$

$$1 = (\Lambda^0{}_0)^2 - \sum_{i=1}^{3} (\Lambda^0{}_i)^2$$

$$\hookrightarrow (\Lambda^0{}_0)^2 \geq 1$$

Thus, $\det \Lambda$ and $\operatorname{sgn} \Lambda^0{}_0$ must be constant on any one component.

The four possibilities are

$L^\uparrow_+$: $\det \Lambda = 1$, $\operatorname{sgn} \Lambda^0{}_0 = 1$, which contain 1

$L^\uparrow_-$: $\det \Lambda = -1$, $\operatorname{sgn} \Lambda^0{}_0 = 1$, which contain I_s

$L^\downarrow_+$: $\det \Lambda = 1$, $\operatorname{sgn} \Lambda^0{}_0 = -1$, which contain I_{st}

$L^\downarrow_-$: $\det \Lambda = -1$, $\operatorname{sgn} \Lambda^0{}_0 = -1$, which contain I_t

The Lorentz transformation

I_s (space inversion): 空间反射

$$(I_s x)^0 = x^0 \quad ; \quad (I_s x)^i = -x^i \quad i = 1,2,3$$

I_t (time inversion): 时间反演

$$(I_t x)^0 = -x^0 \ , \quad (I_t x)^i = x^i \quad i = 1,2,3$$

I_{st} (space-time inversion): 时空反演

$$(I_{st}\, x) = -x = (I_s I_t\, x)$$

Clearly,

$$L_-^{\uparrow} \xrightarrow{I_s} L_+^{\uparrow}$$

$$L_-^{\downarrow} \xrightarrow{I_t} L_+^{\uparrow}$$

$$L_+^{\downarrow} \xrightarrow{I_{st}} L_+^{\uparrow}$$

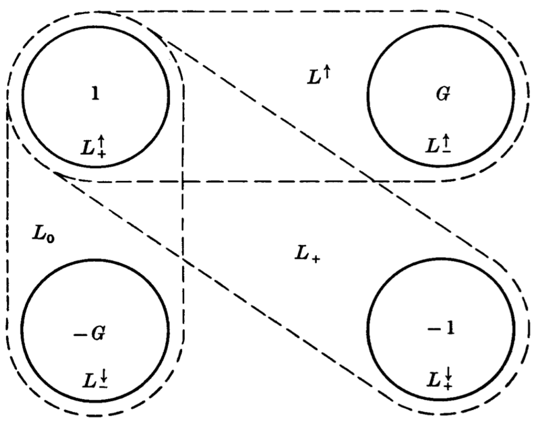


FIGURE 1–1. Connectivity properties of the Lorentz group, L, and its subgroups: the proper Lorentz group, L_+; the orthochronous Lorentz, $L^\uparrow$; the orthochorous Lorentz group, L_0; and the restricted Lorentz group, $L_+^\uparrow$.

The important subgroups of L:

$L^\uparrow = L_+^\uparrow \cup L_-^\uparrow$ the orthochronous Lorentz group 正时 $\Lambda^0{}_0 \geq +1$

$L_+ = L_+^\uparrow \cup L_+^\downarrow$ the proper Lorentz group 正常 $\det\Lambda = +1$

$L_0 = L_+^\uparrow \cup L_-^\downarrow$ the orthochorous Lorentz group 正统 $\operatorname{sgn}\Lambda^0{}_0 \det\Lambda = 1$

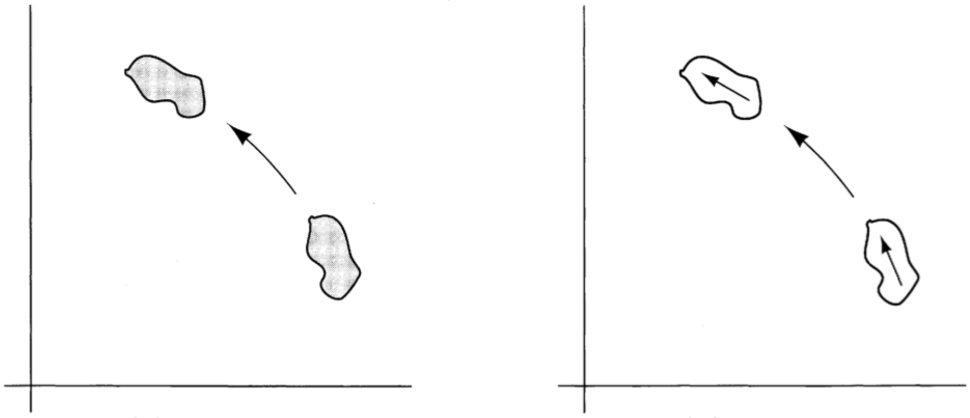
$L_+^\uparrow$ the proper orthochronous Lorentz group

or the restricted Lorentz group

Classical field theory:

Scalar field: $\phi(x) \xrightarrow{\Lambda} \phi'(x) = \phi(\Lambda^{-1}x)$

Vector field: $A^\mu(x) \xrightarrow{\Lambda} \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$



(a) scalar field (b) vector field

Figure 3.1. When a rotation is performed on a vector field, it affects the *orientation* of the vector as well as the location of the region containing the configuration.

E.g. The Klein-Gorden field

$$\partial_\mu\phi(x) \longrightarrow \partial_\mu\phi(\Lambda^{-1}x) = (\Lambda^{-1})^\nu{}_\mu [\partial_\nu\phi](\Lambda^{-1}x)$$
$$= \Lambda_\mu{}^\nu (\partial_\nu\phi)(\Lambda^{-1}x)$$

$$\boxed{\begin{aligned} \frac{\partial}{\partial x^\mu} &= \frac{\partial}{\partial y^\nu}\frac{\partial y^\nu}{\partial x^\mu} \qquad y^\nu = (\Lambda^{-1})^\nu{}_\alpha x^\alpha \\ &= \frac{\partial}{\partial y^\nu}(\Lambda^{-1})^\nu{}_\mu \end{aligned}}$$

The derivate terms in the Lagrangian transform as

$$\mathcal{L}_{deriv}(x) = \partial_\mu\phi(x)\,\partial_\nu\phi(x)\,g^{\mu\nu} \longrightarrow \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma [\partial_\rho\phi](y)\,[\partial_\sigma\phi](y)\,g^{\mu\nu}$$
$$= g^{\rho\sigma}[\partial_\rho\phi](y)\,[\partial_\sigma\phi](y)$$
$$= \mathcal{L}_{deriv}(y)$$

The potential terms transform in the same way $\phi^2(x) \longrightarrow \phi^2(y)$, then

$$S = \int d^4x\, \mathcal{L}(x) \longrightarrow \int d^4x\, \mathcal{L}(y) = \int d^4y \left|\frac{\partial x}{\partial y}\right| \mathcal{L}(y)$$
$$= \int d^4y\, \mathcal{L}(y) \qquad \leftarrow |\det\Lambda| = 1$$

Quantum Lorentz Transformation & The Poincaré Algebra

For an infinitesimal inhomogeneous Lorentz transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \quad , \quad a^\mu = \varepsilon^\mu$$

Then we have

$$g_{\mu\nu} = g_{\rho\sigma} (\delta^\rho{}_\mu + \omega^\rho{}_\mu) (\delta^\sigma{}_\nu + \omega^\sigma{}_\nu)$$

$$= g_{\mu\nu} + \underbrace{\omega_{\mu\nu}}_{\equiv g_{\rho\mu}\omega^\rho{}_\nu} + \omega_{\nu\mu} + O(\omega^2)$$

↳ $\omega_{\mu\nu} = - \omega_{\nu\mu}$ antisymmetry . 6 independent components

+ 4 components of ε^μ

↳ An Poincaré transformation is described by $6 + 4 = 10$ parameters

The transformation $T(\Lambda, a)$ induce a unitary linear transformation in the Hilbert space.

$$|\Psi\rangle \rightarrow U(\Lambda, a) |\Psi\rangle$$

The operators U satisfy

$$U(\bar\Lambda, \bar a) U(\Lambda, a) = U(\bar\Lambda\Lambda, \bar\Lambda a + \bar a)$$

Choose $U(1, 0) = \mathbb{1}$ unit operator

Then

$$U(1+\omega, \varepsilon) = \mathbb{1} + \frac{1}{2} i \omega_{\rho\sigma} J^{\rho\sigma} - i \varepsilon_\rho P^\rho + \cdots$$

ω, ε independent operators

Since $U(1+\omega, \varepsilon)$ be unitary , $J^{\rho\sigma}$ and P^ρ must be Hermitian

$$J^{\rho\sigma\dagger} = J^{\rho\sigma} \quad , \quad P^{\rho\dagger} = P^\rho$$

↑
since ω is antisymmetric
we choose $J^{\rho\sigma} = - J^{\sigma\rho}$

(P^0 : Hamiltonian
P^i : momentum operator
J^{ij} : angular momentum
$i, j = 1, 2, 3$)

Lorentz transformation of $J^{\rho\sigma}$ and P^{ρ}

$$U(\Lambda, a)\, J^{\rho\sigma}\, U^{-1}(\Lambda, a) = \Lambda_{\mu}{}^{\rho} \Lambda_{\nu}{}^{\rho} (J^{\mu\nu} - a^{\mu} P^{\nu} + a^{\nu} P^{\mu})$$

$$U(\Lambda, a)\, P^{\rho}\, U^{-1}(\Lambda, a) = \Lambda_{\mu}{}^{\rho}\, P^{\mu}$$

Then if $a^{\mu} = 0$. $J^{\mu\nu}$ is a tensor , P^{μ} is a vector

if $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$ (pure translations) P^{ρ} is invariant , but $J^{\rho\sigma}$ is not

Lie algebra of the Poincaré group.

$$i [J^{\mu\nu}, J^{\rho\sigma}] = g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\sigma\mu} J^{\mu\nu} + g^{\sigma\nu} J^{\rho\mu}$$

$$i [P^{\mu}, J^{\rho\sigma}] = g^{\mu\rho} P^{\sigma} - g^{\mu\sigma} P^{\rho}$$

$$[P^{\mu}, P^{\rho}] = 0$$

conserved operators (commute with $P^{0} = H$)

$$P = \{ P^{1}, P^{2}, P^{3} \} \qquad J = \{ J^{23}, J^{31}, J^{12} \} \qquad P^{0}$$

not conserved operators

$$K = \{ J^{10}, J^{20}, J^{30} \}$$ (do not use K to label physical state)

Pure translation $T(1, a)$: subgroup of $T(\Lambda, a)$

$$T(1, \bar{a})\, T(1, a) = T(1, \bar{a} + a)$$

$$U(1, a) = e^{-i P^{\mu} a_{\mu}}$$

space-time translation on the field operator $\hat{\phi}$

$$e^{-i P\cdot x}\, \hat{\phi}(0)\, e^{i P\cdot x} = \hat{\phi}(x)$$

A rotation R_{θ} by an angle $|\theta|$ around the direction of $\vec{\theta}$

$$U(R_{\theta}, 0) = e^{i \vec{J}\cdot\vec{\theta}}$$