2. The quantized scalar field.

Construction principles for a Lagrangian.

- Specify the fields the theory should contain & the symmetries under which it should be invariant.

- Build $\mathcal{L}$ as a sum of products of the field

$$\mathcal{L} = \sum_i g_i \, O_i[\phi]$$

such that S is a Lorentz scalar and invariant under all other postulated symmetries. Include all such terms in $\mathcal{L}$ up to mass dimension 4 or a specified number > 4.

- $\mathcal{L}$ must contain derivatives of ϕ : $\partial_\mu \phi$. Otherwise there would be no dynamics, i.e. $\pi = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = 0$ and $\phi(x)$ is not a physical d.o.f.

2.1 The real (hermitian) scalar fields.

$\partial^\mu\phi\,\partial_\mu\phi$ is the simplest Lorentz scalar, which contains $\partial_\mu\phi$

$\hookrightarrow \quad \mathcal{L} \supset \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi$

$\hookrightarrow \quad \phi$ has mass dimension 1, $[\partial_\mu] = [\frac{1}{x}] = 1$

The most general Lagrangian is of the form

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi)$$

- A Lagrangian of this form describes the Higgs field in the SM
- V, the potential term, does not depend on $\partial_\mu\phi$. One could add higher order kinetic terms such as $K_1(\partial_\mu\partial_\nu\phi)(\partial^\mu\partial^\nu\phi)\cdots$, but this is admissible only if the theory is interpreted as an effective QFT
- Usually $V(\phi)$ is a polynomial in ϕ:

$$V(\phi) = \lambda_0 + \lambda_1\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda_3}{3!}\phi^3 + \frac{\lambda_4}{4!}\phi^4 + \cdots$$

λ_0 is irrelavant for the dynamics, since it drops out in the E.L. eq.

λ_1 can be removed by the field redefinition $\phi = \phi' - \lambda_1/m^2$.

$[m] = 1$, $[\lambda_3] = 1$, $[\lambda_4] = 0$

The canonically conjugated field is $\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \dot{\phi}$

The E.L. equation reads $\quad \partial_\mu\partial^\mu\phi + \frac{dV}{d\phi} = 0$

$$\text{or} \quad (\partial^2 + m^2)\phi + \underbrace{\frac{\lambda_3}{2!}\phi^2 + \frac{\lambda_4}{3!}\phi^3}_{\text{non-linear terms, (self)-interaction.}} = 0$$

2.2 Free field, creation and annihilation operators

A field theory is called free, if $\mathcal{L}$ contains only bilinear field products.

Here $\mathcal{L}_0 = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m^2\phi^2 \quad \Rightarrow \quad (\partial^2 + m^2)\phi = 0$

Taking the Fourier transform

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\, \tilde{\phi}(\vec{p},t)$$

Then $\tilde{\phi}(\vec{p},t)$ satisfies

$$(\partial^2 + m^2)\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\, \tilde{\phi}(\vec{p},t)$$

$$= \int \frac{d^3p}{(2\pi)^3}\left[\frac{\partial^2}{\partial t^2} + |\vec{p}|^2 + m^2\right] e^{i\vec{p}\cdot\vec{x}}\, \tilde{\phi}(\vec{p},t) = 0$$

For each value of $\vec{p}$, $\tilde{\phi}(\vec{p},t)$ solves the equation of a harmonic oscillator vibrating at frequency $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$

The general real solution to the KG equation may be expressed as a linear superposition of simple harmonic oscillators, each vibrating at a different frequency with a different amplitude.

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2\omega_{\vec{p}}}}\left(e^{-ip\cdot x}a_{\vec{p}} + e^{ip\cdot x}a^*_{\vec{p}}\right)$$

$$p\cdot x = g_{\mu\nu}p^\mu x^\nu = p^0 t - \vec{p}\cdot\vec{x}$$

See: $(\partial^2 + m^2)\phi = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2\omega_{\vec{p}}}}(-p^2 + m^2)\left(e^{-ip\cdot x}a_{\vec{p}} + e^{ip\cdot x}a^*_{\vec{p}}\right) \underset{\uparrow\, p^2 - m^2 = 0}{=} 0$

define quantum fields as integrals over creation and annihilation operators for

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2\omega_{\vec{p}}}}\left(e^{-ip\cdot x}a_{\vec{p}} + e^{ip\cdot x}a^\dagger_{\vec{p}}\right) \qquad (*)$$

$a_{\vec{p}},\ a^*_{\vec{p}}$: C-number $\rightarrow$ $a_{\vec{p}},\ a^\dagger_{\vec{p}}$ q-number

Note that the operators $a_{\vec{p}}, a^\dagger_{\vec{p}}$ are independent of time. $\phi(x)$ is a solution of the K-G. equation provided $p^\mu = (p^0, \vec{p})$ satisfies $p^2 = m^2$ that is $p^0 = \omega_{\vec{p}} = \sqrt{m^2 + |\vec{p}|^2}$.

In the following we use the short-hand a_p but always remember that a_p is labelled by $\vec{p}$, not the four vector p^μ, since p^0 is fixed through $p^0 = \omega_p = \sqrt{m^2 + \vec{p}^2}$

$\phi(x)$, $\pi(x)$ satisfy the canonical commutation relations. For this to be true. a_p, and its adjoint a_p^+ must satisfy

$$[a_p, a^+_{p'}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')$$
$$[a_p, a_{p'}] = [a^+_p, a^+_{p'}] = 0 \qquad (**)$$

Verify that with $(**)$ the field commutation relations are indeed satisfied.

We first compute

$$\underbrace{[\phi(x), \phi(y)]}_{\text{different times } x^0,\, y^0} = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2\omega_p}} \int \frac{d^3\vec{p}'}{(2\pi)^3\sqrt{2\omega_{p'}}} [e^{-ip\cdot x} a_p + e^{ip\cdot x} a_p^+ , e^{-ip'\cdot y} a_{p'} + e^{ip'\cdot y} a^+_{p'}]$$

$$= \cdots \qquad \cdots \left\{ e^{-ip\cdot x + ip'\cdot y} [a_p, a^+_{p'}] + e^{ip\cdot x - ip'\cdot y} \times [a_p^+, a_{p'}] \right\}$$

$$\underset{\text{insert } (**)}{=} \int \frac{d^3\vec{p}}{(2\pi)^3\, 2\omega_p} [e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}]$$

$\equiv \Delta(x-y)$ "The commutator of two fields has a simple structure as a c-number"

$$\Rightarrow [\phi(t, \vec{x}), \phi(t, \vec{y})] = \Delta(x-y)\Big|_{x^0 = y^0 = t}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} [e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}]$$

$$= 0$$

substitute $\vec{p} \to -\vec{p}$ in the 2nd term

$$\Rightarrow [\phi(t, \vec{x}), \pi(t, \vec{y})] = \left[\frac{\partial}{\partial y^0} \Delta(x-y)\right]_{y^0 = x^0}$$

$$= \left\{ \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} [e^{-ip\cdot(x-y)} (+ip^0) - e^{ip\cdot(x-y)} (-ip^0)] \right\}_{y^0 = x^0}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{i}{2} [e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}]$$

$$= i\, \delta^{(3)}(\vec{x}-\vec{y})$$

$[\pi, \pi]_{y^0 = x^0}$ vanishes by similar arguement as $[\phi, \phi]_{y^0 = x^0}$.

We can prove the more general statement $[\phi(x), \phi(y)] = 0$ whenever $(x-y)^2 < 0$

i.e. x, y are space-like separated.

Proof: $$\Delta(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} \left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right]$$

Here $\int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} = \int \frac{d^4\vec{p}}{(2\pi)^3} \delta(p^2 - m^2)\,\theta(p^0)$ is Lorentz invariant.

$\Rightarrow$ $\Delta(x-y)$ is Lorentz-invariant, we can evaluate it in any convenient frame.

Since $(x-y)^2 < 0$, we choose a frame where $x^0 - y^0 = 0$ and

$(x-y)^2 = -|\vec{x} - \vec{y}|^2 < 0$

The statement then follows, since we already computed the equal-time commutator

Measurements at space-like seperated points do not interfere, a consequene of causality.

The Hamiltonian of the free theory

$$H = \int d^3\vec{x}\ [\ \pi(x)\,\dot\phi(x) - \mathcal{L}(x)\,]$$

$$\pi(x) = \dot\phi(x)$$

$$= \int d^3\vec{x}\ \frac{1}{2}\left[\ (\dot\phi)^2 + |\vec\nabla\phi|^2 + m^2\phi^2\ \right]$$

$$\dot\phi = \int \frac{d^3\vec p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\ [\,e^{-ip\cdot x}a_p - e^{ip\cdot x}a_p^\dagger\,]$$

$$(\dot\phi)^2 = \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}(-1)\sqrt{\frac{\omega_p}{2}}\sqrt{\frac{\omega_q}{2}}\Big(a_pa_qe^{-ip\cdot x-iq\cdot x} - a_p^\dagger a_q e^{ip\cdot x-iq\cdot x} - a_pa_q^\dagger e^{-ip\cdot x+iq\cdot x} + a_p^\dagger a_q^\dagger e^{ip\cdot x+iq\cdot x}\Big)$$

$$(\vec\nabla\phi) = \int\frac{d^3\vec p}{(2\pi)^3\sqrt{2\omega_p}}(i\vec p)\,[\,e^{-ip\cdot x}a_p - e^{ip\cdot x}a_p^\dagger\,]$$

$$|\nabla\phi|^2 = \int\frac{d^3\vec p}{(2\pi)^3\sqrt{2\omega_p}}\int\frac{d^3\vec q}{(2\pi)^3\sqrt{2\omega_q}}(-\vec p\cdot\vec q)\Big(a_pa_qe^{-ip\cdot x-iq\cdot x} - a_p^\dagger a_q e^{ip\cdot x-iq\cdot x} - a_pa_q^\dagger e^{-ip\cdot x+iq\cdot x} + a_p^\dagger a_q^\dagger e^{ip\cdot x+iq\cdot x}\Big)$$

$$m^2\phi^2 = \int\frac{d^3\vec p}{(2\pi)^3\sqrt{2\omega_p}}\int\frac{d^3\vec q}{(2\pi)^3\sqrt{2\omega_q}}\Big[a_pa_qe^{-ip\cdot x-iq\cdot x} + a_p^\dagger a_q e^{ip\cdot x-iq\cdot x} + a_pa_q^\dagger e^{-ip\cdot x+iq\cdot x} + a_p^\dagger a_q^\dagger e^{ip\cdot x+iq\cdot x}\Big]$$

$$\boxed{\int d^3\vec x\, e^{i\vec p\cdot\vec x} = (2\pi)^3\delta^{(3)}(\vec p)}$$

$$\dashrightarrow = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\left(-\frac{\omega_p}{2}\right)\Big(a_pa_{-p}e^{-i2\omega_pt} - a_p^\dagger a_p - a_pa_p^\dagger + a_p^\dagger a_{-p}^\dagger e^{2i\omega_pt}\Big)$$

$$+\frac{1}{2}\ \cdots\ \frac{1}{2\omega_p}\Big[a_pa_{-p}e^{-i2\omega_pt}|\vec p|^2 + a_p^\dagger a_p|\vec p|^2 + a_pa_p^\dagger|\vec p|^2 + a_p^\dagger a_{-p}^\dagger e^{i2\omega_pt}|\vec p|^2\Big]$$

$$+\frac{1}{2}\ \cdots\ \frac{1}{2\omega_p}\Big[a_pa_{-p}e^{-i2\omega_pt} + a_p^\dagger a_p + a_pa_p^\dagger + a_p^\dagger a_{-p}^\dagger e^{2i\omega_pt}\Big]m^2$$

$$= \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\Big[a_pa_{-p}e^{-2i\omega_pt}(-\omega_p^2+|\vec p|^2+m^2) + a_p^\dagger a_p(+\omega_p^2+|\vec p|^2+m^2) + a_pa_p^\dagger(\omega_p^2+|\vec p|^2+m^2) + a_p^\dagger a_{-p}^\dagger e^{2i\omega_pt}(-\omega_p^2+|\vec p|^2+m^2)\Big]$$

$$= \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\omega_p\,(a_p^\dagger a_p + a_pa_p^\dagger)$$

$$\omega_p^2 = |\vec p|^2 + m^2$$

$$= \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\omega_p\Big(2a_p^\dagger a_p + [\,a_p, a_p^\dagger\,]\Big)$$

$$= \int\frac{d^3p}{(2\pi)^3}\omega_p\,[\,\underbrace{a_p^\dagger a_p}_{\text{number operator}} + \underbrace{\tfrac{1}{2}(2\pi)^3\delta^{(3)}(0)}_{\text{formally infinite ``zero-point'' energy because infinitely many d.o.f.}}\,]$$

We will simply drop this infinite term.

"Casimir force"

Dropping the zero-point energy is therefore equivalent to moving all creation operators to the left and annihilation operators to the right.

Normal order : all the a_p are to the right of all the $a_p^\dagger$

$$N[\phi(x_1) \cdots \phi(x_n)] \equiv :\phi(x_1) \cdots \phi(x_n):$$

eg. $N(a_p a_k^\dagger a_q) \equiv a_k^\dagger a_p a_q$

So, for the Hamiltonian, we could write as

$$N(H) = :H: = \int \frac{d^3p}{(2\pi)^3} w_p a_p^\dagger a_p$$

2.3 Quanta of the scalar field and particle states.

We need to identify the particle states or quanta of the field. Although we think of elementary particles as localized, often point-like, objects, it is more natural to think of quanta as quata of energy of a particle with definite momentum. For instance, a photon with momentum $\vec{p} = \hbar \vec{k}$ has energy $E_p = |\vec{p}| c$.

To construct the Hilbert space, we assume that there is a state-vector $|0\rangle$ in the Hilbert space, which satisfies

$$a_p |0\rangle = 0 \qquad \forall \vec{p}$$

$|0\rangle$ is called the vacuum state, with $\langle 0|0\rangle = 1$.

Now "define"

$$|p\rangle = \sqrt{2w_p}\, a_p^\dagger |0\rangle$$ [include a relativistically normalization factor See peskin P_{23}]

Up to now, $\vec{p}$ is only the Fourier-conjugate variable to $\vec{x}$ in the solution of the K.G. eq. and carries no physical interpretation. We now prove that the state $|p\rangle$ has momentum $\vec{p}$ as momentum and energy $w_p = \sqrt{m^2 + |\vec{p}|^2}$. This justifies the interpretation of $\vec{p}$ as momentum and of the state $|p\rangle$ as a single quantum of the field ϕ, or a particle excitation with momentum $\vec{p}$ and rest mass m.

The momentum is the Noether charge of translation symetry.

$$p^i = \int d^3x \; \pi \, \partial^i \phi \qquad [\; \partial^i = -\partial_i = -\nabla^i$$

or $$\vec{p} = -\int d^3x \; \pi \, \vec{\nabla} \phi$$

In terms of creation and annihilation operators, the momentum operator is given by

$$\vec{p} = -\int d^3x \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \int \frac{d^3q}{(2\pi)^3 \sqrt{2\omega_q}} (i\vec{q})$$

$$\times \Big[\; a_p a_q e^{-ip\cdot x - iq\cdot x} - a_p^\dagger a_q e^{ip\cdot x - iq\cdot x} - a_p a_q^\dagger e^{-ip\cdot x + iq\cdot x} + a_p^\dagger a_q^\dagger e^{ip\cdot x + iq\cdot x} \Big]$$

$$= -\int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \Big[-\underbrace{\vec{p}\, a_p a_{-p} e^{-i2\omega_p t}}_{\substack{\text{vanishes,} \\ (\text{odd in } \vec{p})}} - \vec{p}\, a_p^\dagger a_p - \vec{p}\, a_p a_p^\dagger - \underbrace{\vec{p}\, a_p^\dagger a_{-p}^\dagger e^{i2\omega_p t}}_{\substack{\text{vanishes} \\ (\text{odd in } \vec{p})}} \Big]$$

$= 2a_p^\dagger a_p +$ commutator term but it vanishes, there is no zero-point three momentum

$$= \int \frac{d^3p}{(2\pi)^3} \; \vec{p} \; a_p^\dagger a_p$$

Note the similarity with H

Now

$$\vec{P} \, |p\rangle = \int \frac{d^3k}{(2\pi)^3} \; \vec{k} \; a_k^\dagger \underbrace{a_k \, a_p^\dagger}_{a_p^\dagger a_k + (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{p})} |0\rangle$$

$$= 0 + \int d^3k \; \delta^{(3)}(\vec{k}-\vec{p}) \; \vec{k} \; a_k^\dagger |0\rangle$$

$$= \vec{p} \; a_p^\dagger |0\rangle = \vec{p} \, |p\rangle$$

Hence $|p\rangle$ is a momentum eigenstate with momentum $\vec{p}$

Also $\quad H |p\rangle = \omega_p |p\rangle$

Hence $|p\rangle$ describes a relativistic particle with mass m, momentum $\vec{p}$ and energy ω_p.

We can create multi-particle states by acting multiple times with $a^\dagger$

$$\text{N particle state}: \quad |p_1, p_2, \cdots, p_N\rangle = a^\dagger_{p_1} a^\dagger_{p_2} \cdots a^\dagger_{p_N} |0\rangle$$

To count how many particles we have of a certain momentum, we introduce the number operator N:

$$N = \int \frac{d^3p}{(2\pi)^3} a^\dagger_p a_p$$

E.g. $|n_p\rangle$ is a state consisting of n_p identical particles with momentum p.

$$|n_p\rangle = \frac{(a^\dagger_p)^{n_p}}{\sqrt{n_p!}} |0\rangle \qquad \boxed{\text{note}: \ |\vec{p}\rangle = |1_p\rangle}$$

it is easy to show that: $N|n_p\rangle = n_p|n_p\rangle$

See:
$$\begin{aligned} N|n_p\rangle &= \int \frac{d^3k}{(2\pi)^3} a^\dagger_k a_k \frac{(a^\dagger_p)^{n_p}}{\sqrt{n_p!}} |0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3} a^\dagger_k \Big(a^\dagger_p a_k + \underbrace{[a_k, a^\dagger_p]}_{(2\pi)^3 \delta^{(3)}(\vec{k}-\vec{p})} \Big) \frac{(a^\dagger_p)^{n_p-1}}{\sqrt{n_p!}} |0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3} a^\dagger_k a^\dagger_p a_k \frac{1}{\sqrt{n_p}} |n_p - 1\rangle + \underbrace{a^\dagger_p \frac{(a^\dagger_p)^{n_p-1}}{\sqrt{n_p!}} |0\rangle}_{= |n_p\rangle} \end{aligned}$$

by commuting a_k to the right until it annihilate on the vacuum.

$$= n_p |n_p\rangle$$

So, N simply count the number of states at momentum p

A multiparticle state, consisting of many particles of different momenta, can be represented as:

$$|n_{p_1}, n_{p_2} \cdots n_{p_m}\rangle = \prod_{i=1}^{m} \frac{(a^\dagger_{p_i})^{n_{p_i}}}{\sqrt{n_{p_i}!}} |0\rangle$$

Then the number operator N acting on this multiparticle state

$$N|n_{p_1}, n_{p_2} \cdots n_{p_m}\rangle = \left(\sum_{i=1}^{m} n_{p_i} \right) |n_{p_1}, n_{p_2} \cdots n_{p_m}\rangle$$

Note that the state is automatically symmetric under interchange of any two particles. Hence, the scalar field describes bosons.

Define the Fock space as the direct sum

$$\mathcal{F} \equiv \bigoplus_{N=0}^{\infty} \mathcal{H}_N$$

$$\left[\quad H_N = \underbrace{H \otimes H \otimes H \cdots \otimes H}_{N \text{ times}} \qquad \begin{array}{l} H = \text{single particle Hilbert space.} \\ H_N = N \text{ particle} \quad \cdots \quad \cdots \end{array} \right]$$

The number operator commutes with the Hamiltonian $[N, H] = 0$, ensuring the particle number is conserved.

When we consider interactions that create and annihilate particles, interactions take us between the different sectors in the Fock space.

Although we refer to the state $|\vec{p}\rangle$ as particles, they are not localized in space.

In QFT, neither $\phi(x)$ or a_p are good operators acting on the Fock space. They are operator valued distributions, rather than fuctions.

One can construct well defined operators by smearing these distribution over space.

E.g. a wavepacket

$$|\psi\rangle = \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \psi(\vec{p}) |\vec{p}\rangle$$

E.g. $\psi(\vec{p}) = e^{-\frac{\vec{p}^2}{2m}}$

[localized in both position & momentum space]

Relativistically normalized momentum states are : $|p\rangle = \sqrt{2\omega_p}\, a_p^\dagger |0\rangle$

These states satisfy $\langle p | q \rangle = (2\pi)^3\, 2\omega_p\, \delta^{(3)}(\vec{p}-\vec{q})$

The completeness relation for the one-particle states is

$$1 = \int \frac{d^3p}{(2\pi)^3\, 2\omega_p} |p\rangle\langle p|$$