3. Interacting fields

Time evolution :

The dynamics of a quantum theory is determined by its Hamiltonian $H(p, q, t)$

The time dependence of the matrix elements of any operator

$$i \frac{d}{dt} \langle \psi | \, O \, | \phi \rangle = \langle \psi | [\, O, \mathcal{H} \,] | \phi \rangle$$

the commutator is carried at time t

For convenience, $\mathcal{H} = M + N$ N, M are hermitian operators

$$i \frac{d}{dt} O = [\, O, M \,] \qquad\qquad i \frac{d}{dt} | \psi(t) \rangle = N \, | \psi(t) \rangle$$

" separate eqs for time evolution of states and operators. "

Schrödinger picture $M = 0$, $N = H$

Then $q_s(t) = q_s(0) = q_s$, $p_s(t) = p_s(0) = p_s$

$$i \frac{d}{dt} | \psi(t) \rangle_s = H(p_s, q_s, t) \, | \psi(t) \rangle_s$$

Define an operator $U(t, t_0)$ —— the time evolution operator

$$| \psi(t) \rangle_s = U(t, t_0) \, | \psi(t_0) \rangle_s$$

- unitarity $U^{-1}(t, t_0) = U^{+}(t, t_0)$ " probability conserving "
- composition law $U(t, t') \, U(t', t_0) = U(t, t_0)$
- U obeys $i \frac{d}{dt} U(t, t_0) = H(p_s, q_s, t) \, U(t, t_0)$ with $U(t, t) = 1$
- initial + composition law $U(t, t_0) = U^{-1}(t_0, t)$
- If H is time-independent then $U(t, t_0) = e^{-iH(p_s, q_s)(t - t_0)}$

Heisenberg picture : $M = H, \quad N = 0$

The states are defined to be time-independent.

We can identify the Heinsenberg states with the Schrödinger states at $t = 0$

$$|\psi(t)\rangle_H = |\psi(0)\rangle_H = |\psi(0)\rangle_S$$

Thus

$$|\psi(0)\rangle_H = U(0, t)\, |\psi(t)\rangle_S$$

The fundamental p and q operators are time dependent

$$q_H(t) = U^{+}(t,0)\; q_H(0)\; U(t,0) = U^{+}(t,0)\; \underset{\substack{\uparrow \\ q_H(0) = q_S(0) = q_S}}{q_S(0)}\; U(t,0)$$

also

$$O_H(t) = U^{+}(t,0)\; O_S(t)\; U(t,0) = U(0,t)\; O_S(t)\; U^{+}(0,t)$$

$$\frac{d}{dt} O_H(t) = i\,[\, H(p_H(t), q_H(t), t),\; O_H(t)\,]$$

Interaction picture

H_0 : free Hamiltonian

V : interaction

$$M = H_0(p, q) + V(p, q, t)$$

$$i \frac{d}{dt} O_I(t) = [O_I, H_0]$$

$$i \frac{d}{dt} |\psi(t)\rangle_I = V_I |\psi(t)\rangle_I \quad : \text{ key eqns}$$

Choose H_0 is time independent

$$q_I(t) = e^{iH_0(p_S, q_S)t} \, q_S(t) \, e^{-iH_0(p_S, q_S, t)}$$

$$\uparrow$$

$$q_S(t) = q_S(0)$$

Change the states

$$|\psi(t)\rangle_I = e^{iH_0(p_S, q_S)t} |\psi(t)\rangle_S$$

The operators

$$O_I(t) = U_0^\dagger(t, 0) \, O_S(t) \, U_0(t, 0) = e^{iH_0(p_S, q_S)t} \, O_S(t) \, e^{-iH_0(p_S, q_S)t}$$

This ensures

$$_I\langle\psi(t)| O_I(t) |\psi(t)\rangle_I = {}_S\langle\psi(t)| O_S(t) |\psi(t)\rangle_S$$

For $|\psi(t)\rangle_I$

$$i \frac{d}{dt} |\psi(t)\rangle_I = V(p_I, q_I, t) |\psi(t)\rangle_I \, , \quad \text{with } V(p_I, q_I, t) \equiv V_I(t)$$

$$= e^{iH_0(p_S, q_S)t} \, V_S(p_S, q_S, t) \, e^{-iH_0(p_S, q_S)t}$$

Note : even V_S is time-independent, V_I can be time-dependent since $p_I(t), q_I(t)$

Introduce $U_I(t, t_0)$

$$|\psi(t)\rangle_I = U_I(t, t_0) |\psi(t_0)\rangle_I$$

- $U_I^{-1}(t, t_0) = U_I^\dagger(t, t_0)$
- $U_I(t, t') U_I(t', t_0) = U_I(t, t_0)$
- $U_I(t, t_0) = U_I^{-1}(t_0, t)$

All pictures coincide at $t=0$

$$|\psi(0)\rangle_H = |\psi(0)\rangle_I = |\psi(0)\rangle_S \qquad O_H(0) = O_I(0) = O_S(0)$$

Then

$$U(t,0)\,|\psi(0)\rangle_S = |\psi(t)\rangle_S$$

$$|\psi(t)\rangle_I = e^{iH_0t}\,|\psi(t)\rangle_S = U_I(t,0)\,|\psi(0)\rangle_I$$

$$\rightarrow \quad U_I(t,0) = e^{iH_0t}\,U(t,0)$$

$$\rightarrow \quad U_I(t,t_0) = U_I(t,0)\,U_I(0,t_0)$$

$$= U_I(t,0)\,U_I^{+}(t_0,0)$$

$$= e^{iH_0t}\,U(t,0)\,U^{+}(t_0,0)\,e^{-iH_0t_0}$$

$$= e^{iH_0t}\,U(t,t_0)\,e^{-iH_0t_0}$$

Finally,

$$i\frac{d}{dt}U_I(t,t_0) = V_I(t)\,U_I(t,t_0) \qquad \text{with} \quad U_I(t_0,t_0) = 1$$

Dyson's formular (proof see Sec. 3.3)

$$U_I(t,t_0) = T\,\exp\left[-i\int_{t_0}^{t}dt'\,V_I(t')\right]$$

time-ordered product

$$T(\,O_1(t_1)\,O_2(t_2)\cdots O_n(t_n)\,) = O_{i_1}(t_{i_1})\,O_{i_2}(t_{i_2})\cdots O_{i_n}(t_{i_n}) \qquad \text{with} \quad t_{i_1} > t_{i_2} > \cdots > t_{i_n}$$

$\boxed{\text{The S-matrix}}$

Consider a non-relativistic Hamiltonian

$$H = \frac{p^2}{2m} + V(\vec{x})$$

$$V(\vec{x}) \xrightarrow{|x| \to \infty} 0 \quad : \text{ no long-range force}$$

In the very far past from $V(x)$, a wave packet moves towards $V(x)$. It goes along as if it were a free wave packet (obey the free Schrödinger eqn) until it intersets $V(x)$. And then, after a while, one looks in the very far future one has new free wave packets moving away from $V(x)$.

In the Schrödinger picture. $|\psi(t)\rangle$ is the solution of free Schrödinger eqn

$$|\psi(t)\rangle = e^{-iH_0(t-t')}|\psi(t')\rangle \equiv U_0(t,t')|\psi(t')\rangle$$

$$H_0 = p^2/(2m)$$

wave packet in the far past

solutions to H_0, belong to a Hilbert space $\mathcal{H}_0$.

.. .. H, ... $\mathcal{H}$.

H_0 is asumed to have the same specturm as H

In $\mathcal{H}$ there is a state $|\psi(t)\rangle^{in}$ in the past, looks like $|\psi(t)\rangle$

$$\lim_{t\to-\infty} \left\| e^{-iH_0 t}|\psi(0)\rangle - e^{-iHt}|\psi(0)\rangle^{in} \right\| = 0$$

Similarly, in the far future,

$$\lim_{t\to+\infty} \left\| e^{-iH_0 t}|\phi(0)\rangle - e^{-iHt}|\phi(0)\rangle^{out} \right\| = 0$$

Think of classical scattering

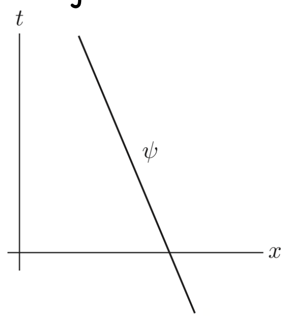


Figure 7.1: The free ψ ($V = 0$)

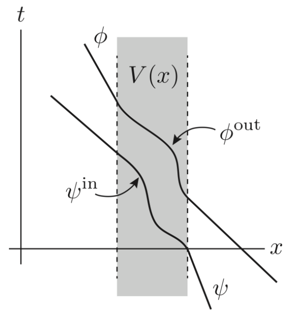


Figure 7.2: ψ^{in} and ϕ^{out}, asymptotic to the free ψ and ϕ

In scattering theory, one wants to known ${}^{\text{out}}\langle \phi | \psi \rangle^{\text{in}}$

Note : don't have to put t since ther inner production is independent of time

$${}^{\text{out}}\langle \phi(t) | \psi(t) \rangle^{\text{in}} = {}^{\text{out}}\langle \phi(t_0) | e^{iH(t-t_0)} e^{-iH(t-t_0)} | \psi(t_0) \rangle^{\text{in}} = {}^{\text{out}}\langle \phi(t_0) | \psi(t_0) \rangle^{\text{in}}$$

Then define an operator in $\mathcal{H}_0$, the scattering matrix S

Wheeler '37
Heisenberg '43

$$\langle \phi | S | \psi \rangle \equiv {}^{\text{out}}\langle \phi | \psi \rangle^{\text{in}}$$

- S-matrix conserves probability $\quad SS^{\dagger} = S^{\dagger}S = 1$
- S-matrix conserves energy $\quad [S, H_0] = 0$

Take V and multiply it by a function $f(t, T, \Delta)$

Then

$$\mathcal{H} = H_0 + V \rightarrow H_0 + f(t, T, \Delta)\, V$$

a so-called adiabatic turning on and off of the interaction

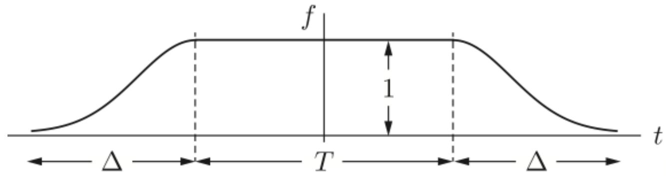


Figure 7.3: The adiabatic function $f(t, T, \Delta)$

Then the in- and out- states are

$$|\psi\rangle^{in} = \lim_{t\to-\infty} e^{iHt} e^{-iH_0 t} |\psi\rangle = \lim_{t\to-\infty} U_I(0, t)|\psi\rangle$$

$$|\phi\rangle^{out} = \lim_{t\to\infty} e^{iHt} e^{-iH_0 t} |\phi\rangle = \lim_{t\to-\infty} U_I(0, t)|\phi\rangle$$

In the far past and far future, $f(t) = 0$ and $\mathcal{H} = \mathcal{H}_0$, the Hamiltonian that gives the evolution of the asymptotically simple state H_0 is the full H. So the S- matrix is written

$$S = \lim_{\substack{T\to\infty \\ \Delta\to\infty \\ \Delta/T\to 0}} U_I(\infty, -\infty)$$

Note : $T \to \infty$ interaction on for a long time

$\Delta \to \infty$ turn V on and off adiabatically

$\Delta/T \to 0$: transient terms are trivial

The interaction picture is suitable

3.1 The Lehmann-Symanzik-Zimmermann reduction formula. [Schwartz 6.1]

The LSZ reduction formula relates S-matrix to the quantum fields

$$\langle f|S|i\rangle = \left[i\int d^4x_1\, e^{-ip_1\cdot x_1} (\partial^2+m^2) \right] \cdots \left[i\int d^4x_n\, e^{ip_n\cdot x_n} (\partial^2+m^2) \right]$$

$$\times \langle \Omega|\, T\{ \phi(x_1)\phi(x_2)\cdots\phi(x_n)\} \,|\Omega\rangle$$

NOTE : 1°. the $-i$ in the exponent applying for initial states ;
" $+i$ " " " " final " "

2°. $T\{\cdots\}$: a time-ordered product

3°. $|\Omega\rangle$ is the vacuum of interacting theory

4°. (∂^2+m^2) vannishes for the asymptotic states, so it removes all terms in the time-ordered product except those with poles of the form $\frac{1}{p^2-m^2}$

5°. to study interacting theories, one assumes

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p(t)\, e^{-ip\cdot x} + a_p^\dagger(t)\, e^{ip\cdot x} \right]$$

this assumption is modified in the presence of bound states

" Operater Product Expansion (OPE) & Factorization theorem "

Proof : $|i\rangle^{in}$: in-state. (free state at $t \to -\infty$ in S-picture)

$$|i\rangle^{in} = \sqrt{2\omega_1}\sqrt{2\omega_2}\ a^\dagger_{p_1}(-\infty)\ a^\dagger_{p_2}(-\infty)\ |\Omega\rangle$$

$|f\rangle^{out}$: out-state (free state at $t \to +\infty$ in S-picture)

$$|f\rangle^{out} = \sqrt{2\omega_3}\cdots\sqrt{2\omega_n}\ a^\dagger_{p_3}(+\infty)\cdots a^\dagger_{p_n}(+\infty)\ |\Omega\rangle$$

Then the S-matrix is

$$\langle f|S|i\rangle = 2^{n/2}\sqrt{\omega_1\omega_2\cdots\omega_n}\ \langle\Omega|\ a_{p_3}(\infty)\cdots a_{p_n}(\infty)\ a^\dagger_{p_1}(-\infty)\ a^\dagger_{p_2}(-\infty)\ |\Omega\rangle \qquad (*)$$

The key formula : $i\int d^4x\, e^{ip\cdot x}(\partial^2 + m^2)\,\phi(x) = \sqrt{2\omega_p}\,[\,a_p(\infty) - a_p(-\infty)\,]$

with $p^\mu = (\omega_p, \vec{p})$

To derive this, we need to assume that the theory is free at $t = \pm\infty$.

Proof : $i\int d^4x\, e^{ip\cdot x}(\partial^2 + m^2)\,\phi(x)$

$$= i\int d^4x\, e^{ip\cdot x}(\partial_0^2 - \partial_i^2 + m^2)\,\phi(x)$$

$$= i\int d^4x\, e^{ip\cdot x}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}(\partial_0^2 + |\vec{k}|^2 + m^2)\,[\,a_k(t)\,e^{-ik\cdot x} + a^\dagger_k(t)\,e^{ik\cdot x}\,]$$

$$= i\int dt\, e^{i\omega_p t}\int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}(\partial_0^2 + |\vec{k}|^2 + m^2)\,[\,a_k(t)\,e^{-i\omega_k t}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})$$
$$+ a^\dagger_k(t)\,e^{i\omega_k t}(2\pi)^3\delta^{(3)}(\vec{p}+\vec{k})\,]$$

$$= i\int dt\, e^{i\omega_p t}(2\omega_p)^{-1/2}(\partial_0^2 + \underbrace{|\vec{p}|^2 + m^2}_{\omega_p^2})\,[\,a_p(t)\,e^{-i\omega_p t} + a^\dagger_{-p}(t)\,e^{i\omega_p t}\,]$$

It's true to show that

$$\partial_0\left\{ e^{i\omega_p t}(i\partial_0 + \omega_p)\,[\,a_p(t)\,e^{-i\omega_p t} + a^\dagger_{-p}(t)\,e^{i\omega_p t}\,]\right\}$$

$$= e^{i\omega_p t}(-\partial_0\omega_p + i\omega_p^2)\,[\cdots] + e^{i\omega_p t}(i\partial_0^2 + \omega_p\partial_0)\,[\cdots]$$

$$= i\,e^{i\omega_p t}(\partial_0^2 + \omega_p^2)\,[\,a_p(t)\,e^{-i\omega_p t} + a^\dagger_{-p}(t)\,e^{i\omega_p t}\,]$$

$$= \int dt\,(2\omega_p)^{-1/2}\,\partial_0\left\{ e^{i\omega_p t}(i\partial_0 + \omega_p)\,[\,a_p(t)\,e^{-i\omega_p t} + a^\dagger_{-p}(t)\,e^{i\omega_p t}\,]\right\}$$

$$= \int dt\,(2\omega_p)^{-1/2}\,\partial_0\left\{ e^{i\omega_p t}(2\omega_p)\,a_p(t)\,e^{-i\omega_p t} + 0\right\}$$

$\uparrow$
$\partial_0\, a_p(t) = 0$ at $t = \pm\infty$

$$= (2\omega_p)^{1/2}\,[\,a_p(+\infty) - a_p(-\infty)\,]$$

by taking the Hermitian conjugate

$$-i\int d^4x\, e^{-ip\cdot x}(\partial^2+m^2)\phi(x) = \sqrt{2\omega_p}\,[\,a_p^\dagger(\infty) - a_p^\dagger(-\infty)\,]$$

In (*), the operators are in time order, so

$$\langle f|S|i\rangle = 2^{n/2}\sqrt{\omega_1\omega_2\cdots\omega_n}\,\langle\Omega|\,T\{a_{p_3}(\infty)\cdots a_{p_n}(\infty)\,a_{p_1}^\dagger(-\infty)\,a_{p_2}^\dagger(-\infty)\}|\Omega\rangle$$

time ordering operator : later are left of earlier

Thus,

$$\langle f|S|i\rangle = 2^{n/2}\sqrt{\omega_1\cdots\omega_n}\,\langle\Omega|\,T\{\,[\,a_{p_3}(\infty) - a_{p_3}(-\infty)]\cdots[\,a_{p_n}(\infty) - a_{p_n}(-\infty)]$$

$$\times\,[\,a_{p_1}^\dagger(\infty) - a_{p_1}^\dagger(\infty)]\,[\,a_{p_2}^\dagger(-\infty) - a_{p_2}^\dagger(-\infty)]\}\,|\Omega\rangle$$

Then we have

$$\langle f|S|i\rangle = [\,i\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial^2+m^2)]\cdots[\,i\int d^4x_n\, e^{ip_n\cdot x_n}(\partial^2+m^2)]$$

$$\times\,\langle\Omega|\,T\{\phi(x_1)\,\phi(x_2)\cdots\phi(x_n)\}\,|\Omega\rangle$$

NOTE: Pulling ∂^2 through $T\{\cdots\}$ is technically not allowed,

3.2 The Feynman propagator [Schwartz 6.2]

The free field operator : $\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{-ik\cdot x} + a_k^\dagger e^{ik\cdot x} \right)$

Then $\langle 0 | \phi(x_1) \phi(x_2) | 0 \rangle = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_1}}} \frac{1}{\sqrt{2\omega_{k_2}}} \underbrace{\langle 0 | a_{k_1} a_{k_2}^\dagger | 0 \rangle}_{= (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2)} e^{ik_2\cdot x_2 - k_1 \cdot x_1}$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{ik\cdot(x_2 - x_1)}$$

We are interested in $\langle 0 | T\{ \phi(x_1) \phi(x_2) \} | 0 \rangle$, so

$$\langle 0 | T\{ \phi(x_1) \phi(x_2) \} | 0 \rangle = \langle 0 | \phi(x_1) \phi(x_2) | 0 \rangle \theta(t_1 - t_2) + \langle 0 | \phi(x_2) \phi(x_1) | 0 \rangle \theta(t_2 - t_1)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[e^{ik\cdot(x_2 - x_1)} \theta(t_1 - t_2) + e^{ik\cdot(x_1 - x_2)} \theta(t_2 - t_1) \right]$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} e^{-i\omega_k \tau} \theta(\tau) + e^{-i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} e^{i\omega_k \tau} \theta(-\tau) \right]$$

$\tau = t_1 - t_2$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} \left[e^{i\omega_k \tau} \theta(-\tau) + e^{-i\omega_k \tau} \theta(\tau) \right]$$

taking $\vec{k} \to -\vec{k}$ in the first term.

Then we need the identity : $\lim_{\varepsilon \to 0} \frac{-2\omega_k}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega^2 - \omega_k^2 + i\varepsilon} e^{i\omega\tau} = e^{-i\omega_k \tau} \theta(\tau) + e^{i\omega_k \tau} \theta(-\tau)$

Proof : $\frac{1}{\omega^2 - \omega_k^2 + i\varepsilon} = \frac{1}{2\omega_k} \left[\frac{1}{\omega - (\omega_k - i\varepsilon)} - \frac{1}{\omega - (-\omega_k + i\varepsilon)} \right]$ (*)

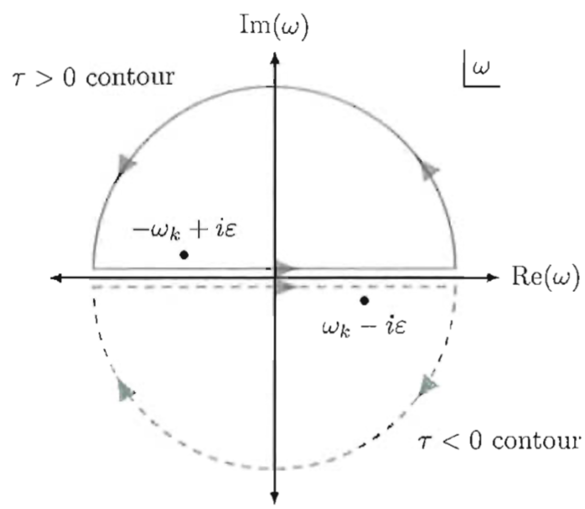


Fig. 6.1 Contour integral for the Feynman propagator. Poles are at $\omega = \pm\omega_k \mp i\varepsilon$. For $\tau > 0$ we close the contour upward, picking up the left pole, for $\tau < 0$ we close the contour downward, picking up the right pole.

Then $$\int_{-\infty}^{+\infty} \frac{d\omega}{\omega-(\omega_k - i\varepsilon)} e^{i\omega\tau} = -2\pi i\, e^{i\omega_k \tau}\,\theta(-\tau)$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{\omega-(-\omega_k + i\varepsilon)} e^{i\omega\tau} = 2\pi i\, e^{-i\omega_k \tau}\,\theta(\tau)$$

$$\Rightarrow \qquad \lim_{\varepsilon\to 0} \frac{-2\omega_k}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega^2-\omega_k^2+i\varepsilon} e^{i\omega\tau} = e^{-i\omega_k\tau}\theta(\tau) + e^{i\omega_k\tau}\theta(-\tau)$$

Putting it together, we find

$$\langle 0|\, T\{\phi(x_1)\,\phi(x_2)\}\,|0\rangle = \lim_{\varepsilon\to 0} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)} \int d\omega\, \frac{-2\omega_k}{2\pi i} \frac{1}{\omega^2-\omega_k^2+i\varepsilon} e^{i\omega\tau}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} e^{ik\cdot(x_1-x_2)}$$

$$\equiv \Delta_F(x_1-x_2) \qquad \text{Feynman propagator}$$

NOTE : 1°. $\Delta_F(x_1-x_2)$ has a pole at $k^2=m^2$, cancelled by the prefactors in the LSZ reduction formula in the projection onto one-particle states.

2° $k^0 \neq \omega_k = \sqrt{|\vec{k}|^2+m^2}$, integration variable.

3° $i\varepsilon$ prescription is just a trick for time ordering

3.3 Perturbation expansion of correlation functions [Peskin 4.2]

We first consider two-point correlation function (two-point Green function)

$$\langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle$$

NOTE : 1°. $\phi(x)$ is not free field

2°. $|\Omega\rangle$ is the ground state of the interacting theory.

We consider the Hamiltonian of ϕ^4 theory as

$$H = H_0 + V = H_{K-G} + \int d^3x \frac{\lambda}{4!}\phi^4$$ (consider H independent on t)

In the Heisenberg picture, the field is $\phi(x) = e^{iHt}\,\underbrace{\phi(\vec{x})}_{\text{field operator in the Schrödinger picture.}}\,e^{-iHt}$ [reference time $t=0$]

We want to express both $\phi(x)$ and $|\Omega\rangle$ in terms of $\phi_0(x)$ and $|0\rangle$

In the above equation, we choose $t=0$ to define the Heisenberg and the Schrödinger picture. More generally, we could have

$$\phi(t,\vec{x}) = e^{iH(t-t_0)}\phi(t_0,\vec{x})\,e^{-iH(t-t_0)}$$ [reference time $t=t_0$]

At any fixed time t_0, we can expand $\phi(t_0,\vec{x})$

$$\phi(t_0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}}\left(a_p e^{i\vec{p}\cdot\vec{x}} + a_p^+ e^{-i\vec{p}\cdot\vec{x}}\right)$$

The field operator in the interaction picture is given by

$$\underbrace{\phi_I(t,\vec{x})}_{\text{free fields, fullfill K-G equations.}} \equiv e^{iH_0(t-t_0)}\phi(t_0,\vec{x})\,e^{-iH_0(t-t_0)}$$

Then

$$\phi_I(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}}\left(a_p e^{-ip\cdot x} + a_p^+ e^{ip\cdot x}\right)\Big|_{x^0=t-t^0}$$

The problem now is to express the full Heisenberg field ϕ in terms of ϕ_I

Formally, $$\phi(t,\vec{x}) = e^{iH(t-t_0)}\,\phi(t_0,\vec{x})\,e^{-iH(t-t_0)}$$
$$= \underbrace{e^{iH(t-t_0)}\,e^{-iH_0(t-t_0)}}_{\equiv\, U_I^{+}(t,t_0)}\,\phi_I(t,\vec{x})\,\underbrace{e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}}_{\equiv\, U_I(t,t_0)}$$

time evolution operator, unitary operator

Obviously, $U(t,t_0)$ fullfills a simple differential equation.

$$i\frac{\partial}{\partial t}U_I(t,t_0) = V_I(t)\,U_I(t,t_0)\ ,\quad U(t_0,t_0) = 1 \qquad (*)$$

with the interaction Hamiltonian in the interaction picture :

$$V_I(t) \equiv e^{iH_0(t-t_0)}\,V\,e^{-iH_0(t-t_0)} = \int d^3x\,\frac{\lambda}{4!}\,\phi_I^4$$

The equation $(*)$ has the formal solution as

$$\boxed{U_I(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t}dt'\,V_I(t')\right]\right\}}$$

Proof:
$$i\frac{\partial}{\partial t}U_I(t,t_0) = e^{iH_0(t-t_0)}(H-H_0)\,e^{-iH(t-t_0)}$$
$$= e^{iH_0(t-t_0)}\,V\,e^{-iH(t-t_0)}$$
$$= e^{iH_0(t-t_0)}\,V\,e^{-iH_0(t-t_0)}\,e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}$$
$$\equiv V_I(t)\,U_I(t,t_0)$$

The solution can be expressed as power series in λ:

$$U_I(t,t_0) = 1 + (-i)\int_{t_0}^{t}dt_1\,V_I(t_1) + (-i)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\,V_I(t_1)\,V_I(t_2) + \cdots$$

See : $$\frac{\partial}{\partial t}U_I(t,t_0) = (-i)\,V_I(t) + (-i)^2\,V_I(t)\int_{t_0}^{t}dt_2\,V_I(t_2) + \cdots$$
$$= -i\,V_I(t)\,U_I(t,t_0)$$

Note that the factors of V_I in the above solution stand in time order.

Therefore one can simplify it using time-ordering operator $T\{\cdots\}$

e.g. The V_I^2 term.

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \; V_I(t_1)\, V_I(t_2) = \frac{1}{2}\int_0^t dt_1 \int_0^t dt_2 \; T\{V_I(t_1)\, V_I(t_2)\}$$

A similar identity holds for the higher terms:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \; V_I(t_1)\cdots V_I(t_n) = \frac{1}{n!}\int_{t_0}^{t} dt_1\, dt_2 \cdots dt_n \; T\{V(t_1)\cdots V(t_n)\}$$

Using this identity, we can write $U(t,t_0)$ in a compact form:

$$U_I(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1\, V_I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1\, dt_2 \; T\{V_I(t_1)\, V_I(t_2)\} + \cdots$$

$$\equiv T\left\{ \exp\left[-i\int_{t_0}^{t} dt'\, V_I(t')\right]\right\}$$

When we do real computations we will keep only the first few terms of the series.

It's convenient to generalize the definition of U, allowing its second arguement

$$U_I(t,t') \equiv T\left\{ \exp\left[-i\int_{t'}^{t} dt''\, V_I(t'')\right]\right\} \qquad (t \geq t')$$

It's easy to verify $\quad i\frac{\partial}{\partial t} U_I(t,t') = V_I(t)\, U_I(t,t') \qquad (**)$

Now the initial condition is $U = 1$ for $t = t'$. Then you can show that

$$U_I(t,t') = e^{iH_0(t-t_0)}\, e^{-iH(t-t')}\, e^{-iH_0(t'-t_0)}$$

Proof: From $(*)$, we have $\quad H_I(t) = i\left[\frac{\partial}{\partial t} U_I(t,t_0)\right] U_I^\dagger(t,t_0)$

Then $(**)$ is expressed as

$$i\frac{\partial}{\partial t} U_I(t,t') = i\left[\frac{\partial}{\partial t} U_I(t,t_0)\right] U_I^\dagger(t,t_0)\, U_I(t,t')$$

$$= i\frac{\partial}{\partial t}\left[\underbrace{U_I(t,t_0)\, U_I^\dagger(t,t_0)}_{=1}\, U_I(t,t')\right] - i\, U_I(t,t_0)\frac{\partial}{\partial t}\left[U_I^\dagger(t,t_0)\, U_I(t,t')\right]$$

$$\Rightarrow \frac{\partial}{\partial t}\left[U_I^\dagger(t,t_0)\, U_I(t,t')\right] = 0 \quad \Rightarrow \quad U_I^\dagger(t,t_0)\, U_I(t,t') = f(t_0,t')$$

Since $U_I(t',t') = 1$, $\quad f(t_0,t') = U^\dagger(t',t_0)$

$$\Rightarrow U_I(t,t') = U_I(t,t_0)\, U_I^\dagger(t',t_0)$$

$$= e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}\, e^{iH(t'-t_0)}\, e^{-iH_0(t'-t_0)}$$

$$= e^{iH_0(t-t_0)}\, e^{-iH(t-t')}\, e^{-iH_0(t'-t_0)}$$

NOTE: • $U_I(t,t')$ is unitary $\quad U_I^\dagger(t,t') = U_I^{-1}(t,t') = U_I(t',t)$

• for $t_1 \geq t_2 \geq t_3 \quad U_I(t_1,t_2)\,U_I(t_2,t_3) = U_I(t_1,t_3)$

$$U_I(t_1,t_3)\,[U_I(t_2,t_3)]^\dagger = U_I(t_1,t_2)$$

Relation between $|0\rangle$ and $|\Omega\rangle$

Starting from $|0\rangle$, and evolving with H

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle \qquad \left[\begin{array}{l} E_n \text{ are eigenvalues of } H \\ |n\rangle \text{ are eigenstates of } H \end{array}\right]$$

$$\sum_n |n\rangle\langle n| = 1$$

In the perturbation theory, assume $\langle\Omega|0\rangle \neq 0 \qquad H_0|0\rangle = 0$

Then $$e^{-iHT}|0\rangle = \underbrace{e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle}_{\text{the ground state}} + \underbrace{\sum_{n\neq0} e^{-iE_nT}|n\rangle\langle n|0\rangle}_{\text{other states}}$$

Since $E_n > E_0$ for $n\neq0$, we can get rid of all the $n\neq0$ terms by sending T to ∞ in a slightly imaginary direction: $T \to \infty(1-i\varepsilon)$

Then e^{-iE_0T} dies slowest, and we have

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} (e^{-iE_0T}\langle\Omega|0\rangle)^{-1} e^{-iHT}|0\rangle$$

Since T is very large, we can shift it by a small constant,

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} [e^{-iE_0(T+t_0)}\langle\Omega|0\rangle]^{-1} e^{-iH(T+t_0)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} [e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle]^{-1} e^{-iH(t_0-(-T))}\, e^{-iH_0(-T-t_0)}|0\rangle$$

$$H_0|0\rangle = 0$$

$$= \lim_{T\to\infty(1-i\varepsilon)} [e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle]^{-1} U_I(t_0,-T)|0\rangle \qquad [\text{ Gell-Mann-Low theorem }]$$

Similarly,

$$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|\, U_I(T,t_0)\, (e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle)^{-1}$$

or $|\Omega\rangle^{in} = U_I(-\infty, t_0)|0\rangle$ (see Schwartz (7.53) ; Weinberg (3.1.13))

Let us put together the pieces of the two-point function.

Assume that $x^0 > y^0 > t_0$. Then

$$
\begin{aligned}
\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \lim_{T \to \infty(1-i\varepsilon)} & \left(e^{-iE_0(T-t_0)} \langle 0 | \Omega \rangle \right)^{-1} \langle 0 | U_I(T, t_0) \\
& \times U_I^{\dagger}(x^0, t_0)\, \phi_I(x)\, U_I(x^0, t_0)\, U_I^{\dagger}(y^0, t_0)\, \phi_I(y)\, U_I(y^0, t_0) \\
& \times U_I(t_0, -T) | 0 \rangle \left(e^{-iE_0(t_0-(-T))} \langle \Omega | 0 \rangle \right)^{-1} \\
= \lim_{T \to \infty(1-i\varepsilon)} & \left[|\langle 0 | \Omega \rangle|^2 e^{-iE_0(2T)} \right]^{-1} \\
& \times \langle 0 | U_I(T, x^0)\, \phi_I(x)\, U_I(x^0, y^0)\, \phi_I(y)\, U_I(y^0, -T) | 0 \rangle
\end{aligned}
$$

To get rid of the prefactor, we apply the normalization condition of $|\Omega\rangle$

$$
1 = \langle \Omega | \Omega \rangle = \lim_{T \to \infty(1-i\varepsilon)} \left[|\langle 0 | \Omega \rangle|^2 e^{-iE_0(2T)} \right]^{-1} \langle 0 | U_I(T, -T) | 0 \rangle
$$

For $x^0 > y^0$

$$
\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | U_I(T, x^0)\, \phi_I(x)\, U_I(x^0, y^0)\, \phi_I(y)\, U_I(y^0, -T) | 0 \rangle}{\langle 0 | U_I(T, -T) | 0 \rangle}
$$

in time order

Finally, for any x^0 and y^0

$$
\boxed{\langle \Omega | T[\phi(x) \phi(y)] | \Omega \rangle = \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | T\{ \phi_I(x) \phi_I(y)\, e^{-i\int_{-T}^{T} dt\, V_I(t)} \} | 0 \rangle}{\langle 0 | T\{ e^{-i\int_{-T}^{T} dt\, V_I(t)} \} | 0 \rangle}}
$$

3.4 Wick's theorem and Feynman diagrams [Peskin 4.3 & 4.4]

Consider $\langle 0| T\{\phi_I(x)\, \phi_I(y)\} |0\rangle$

We first decompose ϕ_I into positive and negative frequency parts:

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x)$$

$$\phi_I^+ = \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_p}}\, a_p e^{-ip\cdot x} \qquad \phi_I^- = \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_p}}\, a_p^+ e^{ip\cdot x}$$

$$\phi_I^+(x)|0\rangle = 0 \qquad \langle 0|\phi_I^-(x) = 0$$

Consider the case $x^0 > y^0$. The time-ordered product of two fields is then

$$T\{\phi_I(x)\phi_I(y)\} \underset{x^0>y^0}{=\!=} \phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y)$$

$$= N\{\phi_I(x)\,\phi_I(y)\} + [\phi_I^+(x), \phi_I^-(y)]$$

If $y^0 > x^0$, we have

$$T\{\phi_I(x)\phi_I(y)\} = N\{\phi_I(x)\phi_I(y)\} + [\phi_I^+(y), \phi_I^-(x)]$$

The contraction of two fields is defined as

$$\overbrace{\phi(x)\ \phi(y)} \equiv \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0 \end{cases}$$

[We drop the subscript "I"]

The contraction of two fields is the Feynman propagator.

$$\overbrace{\phi(x)\ \phi(y)} = \Delta_F(x-y) = D_F(x-y)$$

For two fields, we have

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y) + \overbrace{\phi(x)\phi(y)}\}$$

Wick's theorem :

$$T\{\phi(x_1)\phi(x_2)\cdots\phi(x_m)\} = N\{\phi(x_1)\phi(x_2)\cdots\phi(x_m) + \text{all possible contractions}\}$$

E.g. for $m=4$

$$T\{\phi_1\phi_2\phi_3\phi_4\} = N\{\phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$
$$+ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$
$$+ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4\}.$$

with

$$N\{\phi_1\,\phi_2\,\phi_3\,\phi_4\} \quad : \quad D_F(x_1-x_3)\,N\{\phi_2\,\phi_4\}$$

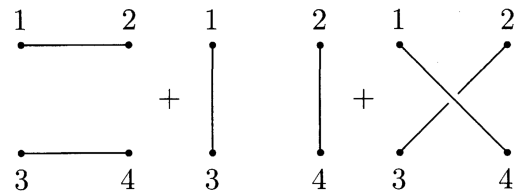
Since $\langle 0|\,N(\text{any operator})\,|0\rangle = 0$, only fully contracted terms survive

Thus,

$$\langle 0|\,T\{\phi_1\,\phi_2\,\phi_3\,\phi_4\}|0\rangle = D_F(x_1-x_2)\,D_F(x_3-x_4) + D_F(x_1-x_3)\,D_F(x_2-x_4)$$
$$+ D_F(x_1-x_4)\,D_F(x_2-x_3)$$

The result can be represented as the sum of three diagrams (Feynman diagrams)

$\langle 0|\,T\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle =$



Now, let's evaluate the two-point function $\langle\Omega|\,T\{\phi(x)\phi(y)\}\,|\Omega\rangle$. We first ignore the denominator

$$\langle\Omega|\,T[\,\phi(x)\,\phi(y)\,]\,|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,T\{\phi_I(x)\,\phi_I(y)\,e^{-i\int_{-T}^{T}dt\,V_I(t)}\}\,|0\rangle}{\langle 0|\,T\{e^{-i\int_{-T}^{T}dt\,V_I(t)}\}\,|0\rangle}$$

The numerator can be expanded as

$$\langle 0|\,T\{\,\phi(x)\,\phi(y) + \phi(x)\,\phi(y)\,[-i\int dt\,V_I(t)] + \cdots\}|0\rangle$$

The second term in ϕ^4 theory is

$$\langle 0|\,T\{\,\phi(x)\,\phi(y)\,(\tfrac{-i\lambda}{4!})\int d^4z\,\phi(z)\,\phi(z)\,\phi(z)\,\phi(z)\,\}\,|0\rangle$$

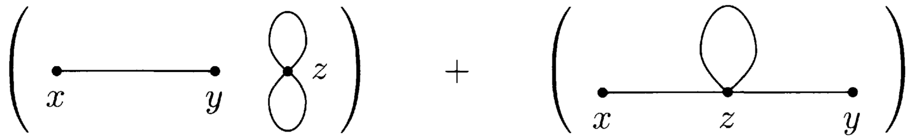
Now apply Wick's theorem. There are $3+12$ ways to contract six ϕ operators.

$\phi(x)\,\phi(y)\,\phi(z)\,\phi(z)\,\phi(z)\,\phi(z)$ $\qquad$ $\phi(x)\,\phi(y)\,\phi(z)\,\phi(z)\,\phi(z)\,\phi(z)$

3 $\qquad$ $4\times 3 = 12$

Thus, $\langle 0 | T\{ \phi(x)\phi(y) \left(\frac{-i\lambda}{4!}\right) \int d^4z\, \phi^4(z) \} | 0 \rangle$

$$= 3 \cdot \left(\frac{-i\lambda}{4!}\right) D_F(x-y) \int d^4z\, D_F(z-z)\, D_F(z-z) \;+\; 12 \cdot \left(\frac{-i\lambda}{4!}\right) \int d^4z\, D_F(x-z)\, D_F(y-z)\, D_F(z-z)$$



External points : x , y

Internal points : z associated with $(-i\lambda/4!)$; vertices

The denominator is

$$\langle 0 | T\{ e^{-i\int dt\, V_I(t)} \} | 0 \rangle = 1 + \langle 0 | T\{ \left(\frac{-i\lambda}{4!}\right) \int d^4z\, \phi^4(z) \} | 0 \rangle + \cdots$$

$$= 1 + 3 \cdot \left(\frac{-i\lambda}{4!}\right) \int d^4z\, D_F(z-z)\, D_F(z-z) + \cdots$$

Therefore, the contribution from the disconnected diagramms cancel between the numerator and the denominator.

In practice, one always draws the diagram first, then writes down the analytic expression.

What is the overall factors ?

The generic vertex has four lines coming in from four different places, so the various placements of these contractions into ϕ^4 generates a factor of $4!$, which cancels the denominator in $(-i\lambda/4!)$. Therefore, it is convenient to associate $\int d^4z\,(-i\lambda)$ with each vertex.

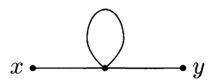
$\phi\;\phi\;\phi\;\phi$ $\quad 4 \times 3 \times 2$ placement of contractions

Symmetry factors : To get the correct overall factor for a diagram, one needs to divide by its symmetry factor, which is in general the number of ways of interchanging components with out changing the diagram.

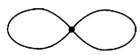
E.g. • the interchange of the two ends of a line.

• the interchange of two lines

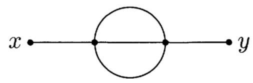
• the equivalence of two vertices



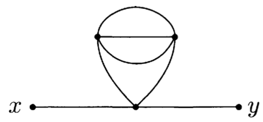
$S = 2$



$S = 2 \cdot 2 \cdot 2 = 8$

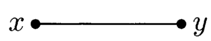


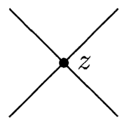
$S = 3! = 6$

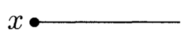


$S = 3! \cdot 2 = 12$

In ϕ^4 - theory the Feynman reules are (position space)

1. For each propagator,  $= D_F(x-y);$

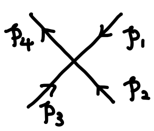
2. For each vertex,  $= (-i\lambda)\int d^4z;$

3. For each external point,  $= 1;$

4. Divide by the symmetry factor.

NOTE : The integral $\int d^4z$ indicates sum of all points where the process can happen.

What happened to the large time $T \to \infty(1-i\epsilon)$?



$$\propto \int d^4z\, e^{-ip_1\cdot z - ip_2\cdot z - ip_3\cdot z + ip_4\cdot z} = \lim_{T\to\infty(1-i\epsilon)} \int_{-T}^{T} dz^0 \int d^3z\, e^{-i(p_1+p_2+p_3-p_4)\cdot z}$$

The exponent blows up as $z^0 \to \infty$ or $-\infty$, unless we take p^0 to have a small imaginary part $p^0 \propto (1+i\epsilon)$. This is the same as the Feynman boundary condition.

$$e^{iz^0p^0} = e^{i|z^0|(1-i\epsilon)|p^0|(1+i\epsilon)} = e^{i|z^0||p^0|(1-\epsilon^2)} = e^{i|z^0||p^0|}$$

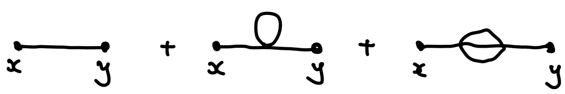
divergence disappears.

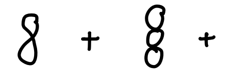
$$\langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle = \lim_{T \to \infty(1-i\varepsilon)} \frac{\langle 0 | T\{\phi_I(x)\phi_I(y)\, e^{-i\int_{-T}^{T} dt\, V_I(t)}\} | 0 \rangle}{\langle 0 | T\{ e^{-i\int_{-T}^{T} dt\, V_I(t)}\} | 0 \rangle}$$

Numerator :

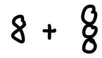
The sum of all diagrams is equal to the sum of all connected diagrams, times the exponential of the sum of all disconnected diagrams.

$$\lim_{T \to \infty(1-i\varepsilon)} \langle 0 | T\{\phi_I(x)\phi_I(y) \exp[-i\int_{-T}^{T} dt\, V_I(t)]\} | 0 \rangle$$

= ( + ⋯)

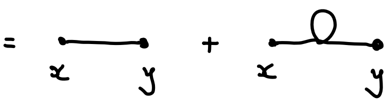
× exp [ + ⋯]

Denominator :

$\langle 0 | T\{\exp[-i\int_{-T}^{T} dt\, V_I(t)]\} | 0 \rangle = \exp[$  $+ \cdots]$

Finally,

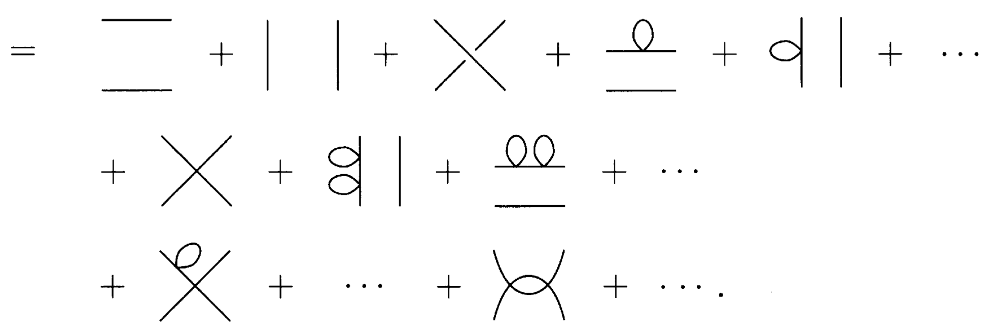
$\langle \Omega | T[\phi(x)\phi(y)] | \Omega \rangle =$ sum of all connected diagrams with two external points

=  + ⋯ .

The generalization to higher correlation functions is easy :

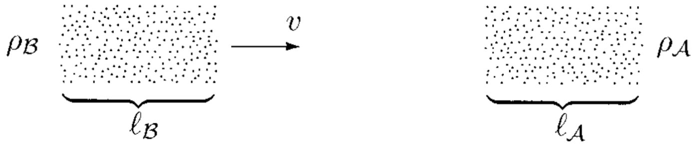
$\langle \Omega | T[\phi(x_1) \cdots \phi(x_n)] | \Omega \rangle = \left(\begin{array}{l} \text{sum of all connected diagramms} \\ \text{with } n \text{ external points} \end{array} \right)$

$$\langle \Omega | T\phi_1\phi_2\phi_3\phi_4 | \Omega \rangle$$



3.5 Cross sections and the S-matrix [Peskin 4.5]

In almost all experiments, we scatter two objects



A : common cross sectional area (area of impact)

The number of scattering events $\propto$ ρ_A, ℓ_A, ρ_B, ℓ_B and A.

Then define the cross section σ as

$$\sigma = \frac{\#\text{ of scatt. events}}{\rho_A \ell_A \rho_B \ell_B A}$$

Note the dimension of σ is $[\sigma] = \frac{1}{[(L/L^3)(L/L^3)L^2]} = L^2$, i.e. area

The simplest way to define the cross section is to define it as the effective size of each particle in the target:

Cross section = Effective size of target particle

Cross sections are often measured in terms of "barns" (1 barn = $10^{-24}\,cm^2$.)

A nucleon is about one fermi, or 10^{-13} cm. Its area is $10^{-26}\,cm^2$ or 0.01 barns.

In classical theory, the cross section is

$$\sigma = \frac{\#\text{ of particles scattered per time}}{\text{flux}} = \frac{\Delta N/\Delta t}{\Phi} = \frac{\Delta N/\Delta t}{\Delta N_{in}/(\Delta t\, A)} = \frac{\Delta N}{N_B}$$

Here N_B is the number of incident particles per area.

The flux $\Phi = \frac{\Delta N_{in}}{\Delta t\ A} = \frac{\rho_B (v_B \Delta t) A}{\Delta t\ A} = \rho_B v_B$

For N_A targets, $N_A = \rho_A \ell_A A$. Then to find the cross section (defined per target) we must divide by N_A, getting

$$\sigma = \frac{\Delta N / \Delta t}{\Phi \; N_A} = \frac{\Delta N}{(v_B \Delta t)\, \rho_B \,(\rho_A \ell_A A)} = \frac{\Delta N}{\rho_B \ell_B \rho_A \ell_A A}$$

Note : if both the target and projectiles are moving, v refers to the relative velocity $v = |\vec{v}_A - \vec{v}_B|$

A more useful quantity is the differential cross section $\dfrac{d\sigma}{d^3p_1 \cdots d^3p_n}$

Decay rate

Consider the decay of an unstable particle at rest

$$\Gamma = \frac{\#\text{ of decay per time}}{\#\text{ of particle}} = \frac{\Delta N}{N \Delta t}$$

In QFT, the propagator is modified as

$$\frac{1}{p^2 - m^2 - im\Gamma}$$

Cross sections from $\mathcal{M}$

Consider a wave packet

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \phi(k) \, |k\rangle$$

Fourier transform of the spatial wave function

$|k\rangle$: one-particle state in the interaction theory

Normalization: $\langle \phi | \phi \rangle = 1, \qquad \int \frac{d^3k}{(2\pi)^3} |\phi(k)|^2 = 1$

The scattering probability:

$$P = \left| {}^{out}\langle \phi_1 \phi_2 \cdots | \phi_A \phi_B \rangle^{in} \right|^2$$

Considering the spatial translation between ϕ_A and ϕ_B

The initial state:

$$|\phi_A \phi_B\rangle_{in} = \int \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{\phi_A(k)\phi_B(k)\, e^{-i b\cdot k_B}}{\sqrt{(2\omega_A)(2\omega_B)}} |k_A k_B\rangle_{in}$$

Note: • $e^{-i\vec{k}_B\cdot\vec{b}}$ can be absorbed into ϕ_B

• $\vec{b}$ is perpendicular to the collision direction

The S-matrix

$${}^{out}\langle p_1 p_2 \cdots | k_A k_B \rangle^{in} \equiv \langle p_1 p_2 \cdots | S | k_A k_B \rangle$$

Define the T-matrix (to isolate the interaction)

$$S \equiv 1 + iT$$

Define the invariant matrix element $\mathcal{M}$

$$\langle p_1 p_2 \cdots | iT | k_A k_B \rangle = (2\pi)^4 \delta^{(4)}(k_A + k_B - \textstyle\sum_f p_f) \; i\mathcal{M}(k_A k_B \to p_1 p_2 \cdots)$$

4 - momentum conservation

Calculate the propability for $|\phi_A \phi_B\rangle$ to $|p_1 p_2 \cdots p_n\rangle$ in $d^3p_1 \cdots d^3p_n$

$$P(AB \to 12\cdots n) = \left(\prod_f \frac{d^3p_f}{(2\pi)^3 2\omega_f}\right) |\,{}^{out}\langle p_1 \cdots p_n | \phi_A \phi_B\rangle^{in}|^2$$

For one target, i.e. $N_A = 1$, the number of scattering events is

$$\Delta N = \int d^2b \; \underset{\substack{\uparrow \\ \text{number density}}}{N_B} \; P(b)$$

Then the cross section (assume N_B is constant)

$$d\sigma = \frac{\Delta N}{N_B} = \int d^2b \; P(b)$$

Replacing the form of $P(b)$, we have

$$d\sigma = \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2\omega_f}\right)\int d^2b \left(\prod_{i=A,B}\int\frac{d^3k_i}{(2\pi)^3}\frac{\phi_i(k_i)}{\sqrt{2\omega_i}}\int\frac{d^3\bar{k}_i}{(2\pi)^3}\frac{\phi_i^*(\bar{k}_i)}{\sqrt{2\omega_i}}\right)$$

$$\times\, e^{ib\cdot(\bar{k}_B - k_B)}\left({}^{out}\langle\{p_f\}|\{k_i\}\rangle^{in}\right)\left({}^{out}\langle\{p_f\}|\{\bar{k}_i\}\rangle^{in}\right)^*$$

Drop the 1 in $1 + iT$ (negelecting forward scattering)

Use

$${}^{out}\langle\{p_f\}|\{k_i\}\rangle^{in} = i\mathcal{M}(\{k_i\}\to\{p_f\})\,(2\pi)^4\,\delta^{(4)}(\Sigma k_i - \Sigma p_f)$$

$${}^{out}\langle\{p_f\}|\{\bar{k}_i\}\rangle^{in} = -i\mathcal{M}^*(\{\bar{k}_i\}\to\{p_f\})\,(2\pi)^4\,\delta^{(4)}(\Sigma \bar{k}_i - \Sigma p_f)$$

$$\int d^2b\, e^{ib\cdot(\bar{k}_B - k_B)} = (2\pi)^2\,\delta^{(2)}(\bar{k}_{B,\perp} - k_{B,\perp})$$

Use $\delta^{(2)}(\bar{k}_{\perp,B} - k_{\perp,B})$ and $\delta^{(4)}(\Sigma\bar{k}_i - \Sigma p_f)$ to perform the $\bar{k}$-integrals

$$\int d\bar{k}_A^z\, d\bar{k}_B^z\, \delta(\bar{k}_A^z + \bar{k}_B^z - \Sigma p_f^z)\,\delta(\bar{\omega}_A + \bar{\omega}_B - \Sigma\omega_f)$$

$$= \int d\bar{k}_A^z\, \delta\left(\sqrt{\bar{k}_A^2 + m_A^2} + \sqrt{\bar{k}_B^2 + m_B^2} - \Sigma\omega_f\right)\Big|_{\bar{k}_B^z = \Sigma p_f^z - \bar{k}_A^z}$$

$$= \frac{1}{|f'(\bar{k}^z_{A,0})|}\Big|_{f(\bar{k}^z_{A,0}) = 0} \qquad \text{with } f = \sqrt{\bar{k}_A^2 + m_A^2} + \sqrt{\bar{k}_B^2 + m_B^2} - \Sigma\omega_f)$$

$$= \frac{1}{\left|\frac{\bar{k}_A^z}{\bar{E}_A} - \frac{\bar{k}_B^z}{\bar{E}_B}\right|} \equiv \frac{1}{\underset{\substack{\uparrow \\ \text{relative velocity in the lab frame.}}}{|v_A - v_B|}}$$

Put everything together

$$d\sigma = \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2\omega_f} \right) \int \frac{d^3 k_A}{(2\pi)^3} \int \frac{d^3 \bar{k}_B}{(2\pi)^3} \frac{|\mathcal{M}(k_A, k_B \to \{p_f\})|^2}{(2\omega_A)(2\omega_B)|v_A - v_B|} |\phi_A(k_A)|^2 |\phi_B(k_B)|^2$$

$$\times (2\pi)^4 \delta^{(4)}(k_A + k_B - \Sigma p_f)$$

For states with wave functions centered on a give momentum $k \to p$

$$d\sigma = \frac{1}{2\omega_A \, 2\omega_B \, |v_A - v_B|} |\mathcal{M}(k_A, k_B \to \{p_f\})|^2 \, d\Pi_f$$

defin $d\Pi_f \equiv (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \Sigma p_f) \prod_f \frac{d^3 p_f}{(2\pi)^3 2E_f}$ [Lorentz invariant phase space]

Note.

- $\omega_A \omega_B |v_A - v_B| = |\omega_B p_A^3 - \omega_A p_B^3| = |\varepsilon_{\mu x y \nu} p_A^\mu p_B^\nu|$

  is not Lorentz invariant.

- if there are n identical particles in the final state, we must either restrict the integration to inequivalent configurations, or divide by $n!$ after integrating over all sets of momenta.

The differential decay rate : ($1 \to n$ process)

$$d\Gamma = \frac{1}{2m_1} |\mathcal{M}|^2 \, d\Pi_n \qquad \text{[define in the rest frame of } p_1 \text{]}$$

Note : an unstable particle cannot be sent into the infinitely distant past.

We will discuss the meaning of this formula latter.

$\boxed{\text{Cross section (another derivation)}}$ [Schwartz 5.1]

The simplest way to define the cross section is to define it as the effective size of each particle in the target:

Cross section = Effective size of target particle

Cross sections are often measured in terms of "barns" (1 barn = $10^{-24}\,cm^2$.)

A nucleon is about one fermi, or 10^{-13} cm. Its area is $10^{-26}\,cm^2$ or 0.01 barns.

In classical theory, the cross section is

$$\sigma = \frac{\#\text{ of particles scattered}}{\text{time} \times \text{flux}} = \frac{N}{T\,\Phi}$$

In QM, the cross section is $d\sigma = \dfrac{dP}{T\,\Phi}$

$d\sigma$ and dP are differential in kinematical variables. (angle or energy)

We first consider the $2 \to n$ process: $p_1 + p_2 \to \sum_j p_j$

In the center of mass frame of p_1 and p_2, $\Phi = \rho \cdot |\vec{v}| = \dfrac{|\vec{v}_1 - \vec{v}_2|}{V}$

Then,

$$d\sigma = \frac{V}{T}\frac{1}{|\vec{v}_1 - \vec{v}_2|}\,dP \qquad [\ \vec{v} = \vec{p}/p^0\ ]$$

the probability dP is

$$dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}\,d\Pi$$

$d\Pi$ is the region of final state momenta

$$d\Pi = \prod_j \frac{V}{(2\pi)^3}\,d^3p_j$$

[We define our states within a box of volume V]

We define $|i\rangle = |p_1\rangle|p_2\rangle$, $|f\rangle = \prod_j |p_j\rangle$

The normalization of the state $|p\rangle$: $\langle p|p\rangle = (2\pi)^3(2\omega_p)\,\delta^{(3)}(0)$

is replaced by $\langle p|p\rangle = 2\omega_p V$ within a box $[\ (2\pi)^3\delta^{(3)}(0) = V\ ]$

Dirac delta $\qquad\qquad$ Kronecker delta

See: $\quad (2\pi)^3\,\delta^{(3)}(\vec{p}) = \int d^3x\, e^{i\vec{p}\cdot\vec{x}} \qquad (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{k}) = \int d^3x\, e^{i(\vec{p}-\vec{k})\cdot\vec{x}} = V\,\delta_{\vec{p}\vec{k}}$

$\Rightarrow (2\pi)^3\,\delta^{(3)}(0) = \int d^3x = V. \qquad \int \frac{d^3p}{(2\pi)^3}\, V\,\delta_{\vec{p}\vec{k}} = 1.$

Similarly. $\quad (2\pi)^4\,\delta^{(4)}(0) = VT$

Then we have $\quad \langle i|i\rangle = (2E_1V)(2E_2V) \quad , \quad \langle f|f\rangle = \prod_j (2E_jV)$

We define transfer matrix T as $\quad S = \underset{\text{NO INTERACTION}}{\underset{\uparrow}{1}} + iT$

It is helpful to factor an overall momentum conservation

$$T = (2\pi)^4\,\delta^{(4)}(p_1+p_2-\sum_f p_f)\,\mathcal{M}$$

Thus we have

$$\langle f|\,S-1\,|i\rangle = (2\pi)^4\,\delta^{(4)}(p_1+p_2-\sum_f p_f)\,\langle f|\mathcal{M}|i\rangle$$

For $|f\rangle \neq |i\rangle$

$$|\langle f|S|i\rangle|^2 = (2\pi)^8\,\delta^{(4)}(p_1+p_2-\sum_f p_f)\,\delta^{(4)}(0)\,|\langle f|\mathcal{M}|i\rangle|^2$$

$$= (2\pi)^4\,\delta^{(4)}(p_1+p_2-\sum_f p_f)\,TV\,|\mathcal{M}|^2$$

$$\cdots |\mathcal{M}|^2 \equiv |\langle f|\mathcal{M}|i\rangle|^2$$

Therefore, the differential probability is

$$dP = \frac{(2\pi)^4\,\delta^{(4)}(p_1+p_2-\sum p_f)\,TV}{(2E_1V)(2E_2V)\prod_f(2E_fV)}\,|\mathcal{M}|^2\,\prod_f \frac{V}{(2\pi)^3}\,d^3p_f$$

$$= \frac{T}{V}\,\frac{1}{(2E_1)(2E_2)}\,|\mathcal{M}|^2\,d\Pi_f$$

with $\quad d\Pi_f \equiv (2\pi)^4\,\delta^{(4)}(p_1+p_2-\sum p_f)\,\prod_f \frac{d^3p_f}{(2\pi)^3\,2E_f} \qquad$ [Lorentz invariant phase space]

Finally, we have

$$d\sigma = \frac{1}{(2E_1)(2E_2)\,|\vec{v}_1-\vec{v}_2|}\,|\mathcal{M}|^2\,d\Pi_f$$

Note. if there are n identical particles in the final state, we must either restrict the integration to inequivalent configurations, or divide by $n!$ after integrating over all sets of momenta.

E.g. for one particle state $p_A + p_B \to p_1$

$$\int d\pi_1 = \int \frac{d^3 p_1}{(2\pi)^3 2E_1} (2\pi)^4 \delta^{(4)}(p_A + p_B - p_1)$$

$$= \int \frac{d^4 p_1}{(2\pi)^3} \delta(p_1^2 - m^2)\, \theta(p_1^0)\, (2\pi)^4 \delta^{(4)}(p_A + p_B - p_1)$$

$$= (2\pi)\, \delta(p_1^2 - m^2)\, \theta(p_1^0)$$

for two particle states $p_A + p_B \to p_1 + p_2$.

$$\int d\pi_2 = \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(p_A + p_B - p_1 - p_2)$$

In the center of mass frame, $\vec{p}_1 + \vec{p}_2 = 0$.

so $$\int d\pi_2 = \int \frac{d^3 p_1}{(2\pi)^3 2E_1 2E_2} (2\pi)\, \delta(E_{cm} - E_1 - E_2)$$ [E_{cm} : total energy]

with $E_1 = \sqrt{m_1^2 + |\vec{p}_1|^2}$, and $E_2 = \sqrt{m_2^2 + |\vec{p}_1|^2}$ ⟵ $\vec{p}_1 + \vec{p}_2 = 0$.

Since $d^3 p_1 = |p_1|^2 d|\vec{p}_1| \sin\theta_1 d\theta_1 d\varphi_1$

$$\int d\pi_2 = \int d|\vec{p}_1| \sin\theta_1 d\theta_1 d\varphi_1 \frac{|\vec{p}_1|^2}{(2\pi)^3 2E_1 2E_2} (2\pi)\, \delta(E_{cm} - E_1 - E_2)$$

$$= \delta\left(E_{cm} - \sqrt{m_1^2 + |\vec{p}_1|^2} - \sqrt{m_2^2 + |\vec{p}_1|^2}\right)$$

$$= \int \sin\theta_1 d\theta_1 d\varphi_1 \frac{1}{16\pi^2} \frac{|\vec{p}_1|}{E_{cm}}$$

3.6 S-matrix and Feynman rules in momentum space [Peskin 4.6 & Schwartz 7.3]

Formula :

$$\langle \vec{p}_1 \cdots \vec{p}_n | \, iT \, | \vec{p}_A \vec{p}_B \rangle$$

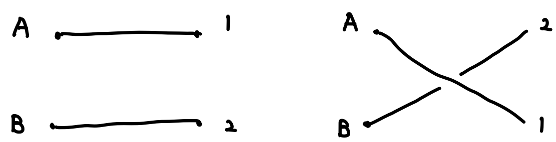
$$= \lim_{T \to \infty(1-i\epsilon)} \left\{ {}_0\langle \vec{p}_1 \cdots \vec{p}_n | \, T\left\{ \exp\left[-i \int_{-T}^{T} dt \, V_I(t) \right] \right\} | \vec{p}_A \vec{p}_B \rangle_0 \right\}_{\text{connected, amputated.}}$$

We first consider the free term.

$${}_0\langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle_0 = \sqrt{(2\omega_1)(2\omega_2)(2\omega_A)(2\omega_B)} \, \langle 0 | \, a_1 a_2 a_A^{+} a_B^{+} | 0 \rangle$$

$$= (2\omega_A)(2\omega_B)(2\pi)^6 \left[\delta^{(3)}(p_A - p_1)\, \delta^{(3)}(p_B - p_2) + \delta^{(3)}(p_A - p_2)\, \delta^{(3)}(p_B - p_1) \right]$$

The corresponding Feynman diagrams



They correspond to the 1 in $S = 1 + iT$, therefore should be excluded

We next consider the first order term in λ,

$${}_0\langle \vec{p}_1 \vec{p}_2 | \, T\left\{ \left(-i \frac{\lambda}{4!}\right) \int d^2x \, \phi^4(x) \right\} | \vec{p}_A \vec{p}_B \rangle_0$$

Consider ϕ^+ acting on an external state:

$$\phi^+(x) | \vec{p} \rangle_0 = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \, a_k e^{ik\cdot x} \sqrt{2\omega_p} \, a_p^{+} | 0 \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \, e^{ik\cdot x} \sqrt{2\omega_p} \left[(2\pi)^3 \delta^{(3)}(k - p) \right] | 0 \rangle$$

$$= e^{ip\cdot x} | 0 \rangle$$

We define the contraction with an external state

$$\overline{\phi(x) \, | \vec{p} \rangle} = e^{ip\cdot x} \qquad\qquad \overline{\langle \vec{p} | \, \phi(x)} = e^{-ip\cdot x}$$

We now analyze the first order term in S-matrix

For $\overline{\phi\phi}\,\overline{\phi\phi}$, we have

$$-i\frac{\lambda}{4!}\int d^4x\ \overline{\phi\phi}\,\overline{\phi\phi}\ \ {}_0\langle p_1\, p_2 | p_A\, p_B\rangle_0$$

which gives vacuum bubble times free term , so it is a disconnected piece. It contributes to 1 not to iT

For $\overline{\phi\phi}\,\phi\phi$. After applying the normal ordering operator , we have

$$a^+a^+ + 2a^+a + aa$$

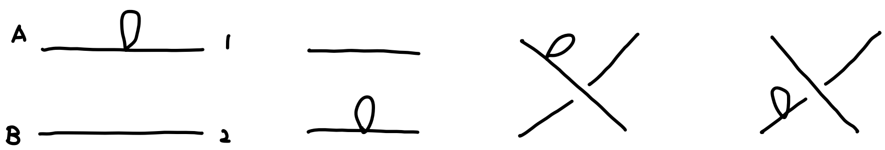
but only the term with the same number of a^+ and a contribute

e.g. $\langle p_1 p_2 | a^+a^+ | p_A p_B\rangle \sim \langle 0 | aa(a^+)^4 | 0\rangle \sim [a, a^+]^2 \langle 0 | a^+a^+ | 0\rangle = 0$

so we have

$$-i\frac{\lambda}{4!}\int d^4x\ \ 2\overline{\phi\phi}\ \ {}_0\langle p_1\, p_2 | \phi^-\phi^+ | p_A\, p_B\rangle_0$$

Doing the contraction with external states , we have



Again , it contributes to 1 , is disconnected.

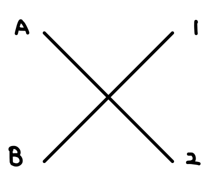
Finally, for only contraction with the exteral states,

$${}_0\langle p_1\, p_2 | \phi^-\phi^-\phi^+\phi^+ | p_A\, p_B\rangle_0$$

Since all ϕs are at the same point, so this gives a term $e^{i(p_A + p_B - p_1 - p_2)\cdot x}$

The total result is

$$4!\ (-i\frac{\lambda}{4!})\int d^4x\ e^{i(p_A + p_B - p_1 - p_2)\cdot x} = -i\lambda\,(2\pi)^4\,\delta^{(4)}(p_A + p_B - p_1 - p_2)$$

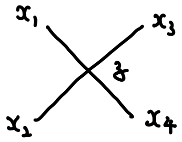


Since $iT = (2\pi)^4\,\delta^{(4)}(p_1 + p_2 - p_A - p_B)\,(i\mathcal{M})$, we have $\mathcal{M} = -\lambda$

Let's apply LSZ formula directly:

We consider $\langle p_3 p_4 | iT | p_1 p_2 \rangle$ or $\langle \Omega | T\{ \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \} | \Omega \rangle$

E.g. the leading - order process [set $m=0$]



$$= \int d^4 z \, (-i\lambda) \, D_F(x_1 - z) \, D_F(x_3 - z) \, D_F(x_2 - z) \, D_F(x_4 - z)$$

$$= (-i\lambda) \int d^4 z \prod_{i=1}^{4} \int \frac{d^4 k_i}{(2\pi)^4} \frac{i}{k_i^2 + i\varepsilon} e^{i k_i \cdot (x_i - z)}$$

Then $[\sigma_{1,2} = +1, \quad \sigma_{3,4} = -1]$

$$i \langle p_3 p_4 | T | p_1 p_2 \rangle = \prod_j \left[-i \int d^4 x_j \, e^{-i p_j \cdot x_j \sigma_j} (\partial_j^2) \right]$$

$$\times (-i\lambda) \int d^4 z \prod_{i=1}^{4} \int \frac{d^4 k_i}{(2\pi)^4} \frac{i}{k_i^2 + i\varepsilon} e^{i k_i \cdot (x_i - z)}$$

$$= (-i\lambda) \int d^4 z \prod_i \int \frac{d^4 k_i}{(2\pi)^4} \frac{i}{k_i^2 + i\varepsilon} \int d^4 x_i \, e^{-i p_i \cdot x_i \sigma_i} (-k_i^2)$$

$$\times e^{i k_i \cdot (x_i - z)}$$

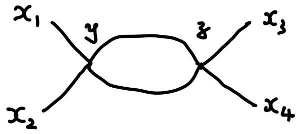
$$= (-i\lambda) \int d^4 z \prod_i \int d^4 k_i \, e^{-i k_i \cdot z} \, \delta^{(4)}(k_i - p_i \sigma_i)$$

$$= (-i\lambda) \int d^4 z \, e^{-i(p_1 + p_2 - p_3 - p_4) \cdot z}$$

$$= (-i\lambda) \underbrace{(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)}_{\text{momentum conservation}}$$

$\Rightarrow \quad i\mathcal{M} = -i\lambda$

E.g. One-loop process



$$\propto (-i\lambda)^2 \int d^4y\, d^4z\; D_F(x_1-y)\, D_F(x_2-y)\, D_F(x_3-z)\, D_F(x_4-z)$$

$$\times\; D_F^2(y-z)^2$$

$$= (-i\lambda)^2 \int d^4y\, d^4z \left[\prod_{i=1}^{2} \int \frac{d^4k_i}{(2\pi)^4} \frac{i}{k_i^2+i\varepsilon}\, e^{ik_i\cdot(x_i-y)} \right]$$

$$\times \left[\prod_{i=3}^{4} \int \frac{d^4k_i}{(2\pi)^4} \frac{i}{k_i^2+i\varepsilon}\, e^{ik_i\cdot(x_i-z)} \right]$$

$$\times \left[\prod_{i=1}^{2} \int \frac{d^4l_i}{(2\pi)^4} \frac{i}{l_i^2+i\varepsilon}\, e^{il_i(y-z)} \right]$$

Then, the T matrix is

$$\langle p_3 p_4 |\, iT\, | p_1 p_2 \rangle$$

$$= \prod_j \left[-i \int d^4x_j\, e^{-ip_j\cdot x_j \sigma_j} (\partial_j^2) \right] \times (-i\lambda)^2 \int d^4y\, d^4z \left[\prod_{i=1}^{2} \int \frac{d^4k_i}{(2\pi)^4} \frac{i}{k_i^2+i\varepsilon}\, e^{ik_i\cdot(x_i-y)} \right]$$

$$\times \left[\prod_{i=3}^{4} \int \frac{d^4k_i}{(2\pi)^4} \frac{i}{k_i^2+i\varepsilon}\, e^{ik_i\cdot(x_i-z)} \right] \times \left[\prod_{i=1}^{2} \int \frac{d^4l_i}{(2\pi)^4} \frac{i}{l_i^2+i\varepsilon}\, e^{il_i(y-z)} \right]$$

$$= (-i\lambda)^2 \int d^4y\, d^4z \left[\prod_{i=1}^{2} \int \frac{d^4k_i}{(2\pi)^4} \frac{i}{k_i^2+i\varepsilon} \int d^4x_i\, e^{-ip_i\cdot x_i\sigma_i} (-k_i^2)\, e^{ik_i\cdot(x_i-y)} \right]$$

$$\times \left[\prod_{i=3}^{4} \int \frac{d^4k_i}{(2\pi)^4} \frac{i}{k_i^2+i\varepsilon} \int d^4x_i\, e^{-ip_i\cdot x_i\sigma_i} (-k_i^2)\, e^{ik_i\cdot(x_i-z)} \right]$$

$$\times \prod_{i=1}^{2} \int \frac{d^4l_i}{(2\pi)^4} \frac{i}{l_i^2+i\varepsilon}\, e^{il_i\cdot(y-z)}$$

$$= (-i\lambda)^2 \int d^4y\, d^4z \left[\prod_{i=1}^{2} \int \frac{d^4k_i}{(2\pi)^4} (-i)(2\pi)^4 \delta^{(4)}(k_i - p_i\sigma_i)\, e^{-ik_i\cdot y} \right]$$

$$\times \left[\prod_{i=3}^{4} \int \frac{d^4k_i}{(2\pi)^4} (-i)(2\pi)^4 \delta^{(4)}(k_i - p_i\sigma_i)\, e^{-ik_i\cdot z} \right]$$

$$\times \prod_{i=1}^{2} \int \frac{d^4l_i}{(2\pi)^4} \frac{i}{l_i^2+i\varepsilon}\, e^{il_i\cdot(y-z)}$$

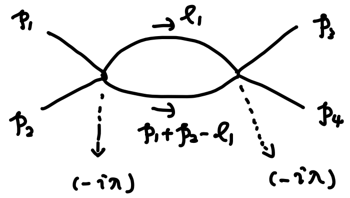
$$= (-i\lambda)^2 \int d^4y\, d^4z\; e^{-ip_1\cdot y - ip_2\cdot y + ip_3\cdot z + ip_3\cdot z} \prod_{i=1}^{2} \int \frac{d^4l_i}{(2\pi)^4} \frac{i}{l_i^2+i\varepsilon}\, e^{il_i\cdot(y-z)}$$

$$= (-i\lambda)^2 \int \frac{d^4l_1}{(2\pi)^4} \frac{i}{l_1^2+i\varepsilon} \int \frac{d^4l_2}{(2\pi)^4} \frac{i}{l_2^2+i\varepsilon} \int d^4y\, e^{-i(p_1+p_2-l_1-l_2)\cdot y} \int d^4z\, e^{i(p_3+p_4-l_1-l_2)\cdot z}$$

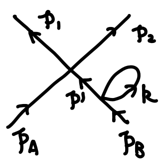
$$= (-i\lambda)^2 \int d^4l_1\, d^4l_2\, \frac{i}{l_1^2+i\varepsilon} \frac{i}{l_2^2+i\varepsilon}\, \delta^{(4)}(p_1+p_2-l_1-l_2)\, \delta^{(4)}(p_3+p_4-l_1-l_2)$$

$$= (-i\lambda)^2 \int \frac{d^4 \ell_1}{(2\pi)^4} \frac{i}{\ell_1^2 + i\varepsilon} \frac{i}{(p_1 + p_2 - \ell_1)^2 + i\varepsilon} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)$$

$$\Rightarrow \quad i\mathcal{M} = (-i\lambda)^2 \int \frac{d^4 \ell_1}{(2\pi)^4} \frac{i}{\ell_1^2 + i\varepsilon} \frac{i}{(p_1 + p_2 - \ell_1)^2 + i\varepsilon}$$



Amputated diagrams



$$= \frac{1}{2} \int \frac{d^4 p'}{(2\pi)^4} \frac{i}{p'^2 - m^2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2} (-i\lambda)(2\pi)^4 (p_A + p' - p_1 - p_2)$$

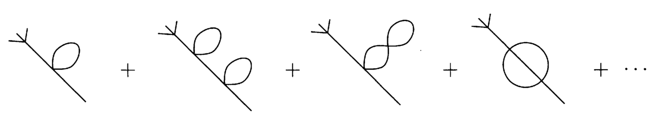
$$\times (-i\lambda)(2\pi)^4 \delta^{(4)}(p_B - p')$$

After integrating over p', we have $\frac{1}{p_B^2 - m^2} = \frac{1}{0}$

p_B is the momentum of an external particle.

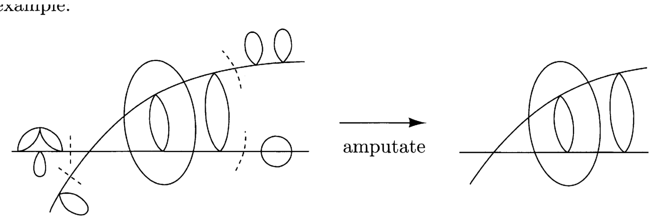
The formula for S-matrix only makes sense after excluding these diagrams.

[proof latter]



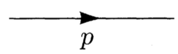
We define amputation as

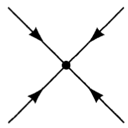
example.



Finally, we have momentum space Feynman rules as

$i\mathcal{M}$ = sum of all connected, amputated diagrams.

1. For each propagator, 

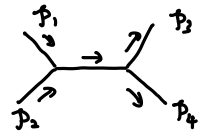
$= \dfrac{i}{p^2 - m^2 + i\epsilon}$;
2. For each vertex,  $= -i\lambda$;
3. For each external line,  $= 1$;
4. Impose momentum conservation at each vertex;
5. Integrate over each undetermined loop momentum: $\displaystyle\int \frac{d^4p}{(2\pi)^4}$;
6. Divide by the symmetry factor.

3.7 Examples

Let's consider ϕ^3 theory

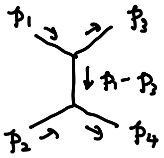
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3$$

At the leading order, there are three diagrams.

$i\mathcal{M}_s =$ 

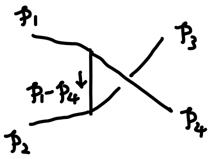
$$= (ig)\frac{i}{(p_1+p_2)^2 - m^2 + i\varepsilon}(ig) = \frac{-ig^2}{s - m^2 + i\varepsilon} \quad [\text{s-channel}]$$

$s = (p_1+p_2)^2$

$i\mathcal{M}_t =$ 

$$= (ig)\frac{i}{(p_1-p_3)^2 - m^2 + i\varepsilon}(ig) = \frac{-ig^2}{t - m^2 + i\varepsilon} \quad [\text{t-channel}]$$

$t = (p_1 - p_3)^2$

$i\mathcal{M}_u =$ 

$$= (ig)\frac{i}{(p_1-p_4)^2 - m^2 + i\varepsilon}(ig) = \frac{-ig^2}{u - m^2 + i\varepsilon} \quad [\text{u-channel}]$$

$u = (p_1 - p_4)^2$

Then the differential cross section

$$\frac{d\sigma}{d\Omega}(\phi\phi \to \phi\phi) = \frac{g^4}{64\pi^2 E_{cm}^2}\left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]$$

The variables s, t and u are called Mandelstam variables. For any $2 \to 2$ processes with initial momenta p_1 and p_2, final momenta p_3 and p_4.

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t \equiv (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u \equiv (p_1 - p_4)^2 = (p_2 - p_4)^2$$

They satisfy

$$s + t + u = 4m^2$$

See: $(p_1 + p_2 - p_3 - p_4)^2 = 0.$

$$\Rightarrow 4m^2 + 2p_1\cdot p_2 - 2p_1\cdot p_3 - 2p_2\cdot p_3 - 2p_1\cdot p_4 - 2p_2\cdot p_3 - 2p_2\cdot p_4 + 2p_3\cdot p_4 = 0$$

$$\Rightarrow \quad (p_1+p_2)^2 + (p_3+p_4)^2 + (p_1-p_3)^2 + (p_2-p_3)^2 + (p_1-p_4)^2 + (p_2-p_3)^2 + (p_2-p_4)^2 = 8m^2$$

$$\Rightarrow \quad s + t + u = 4m^2.$$

In the s-channel, the intermediate state $s > 0$. The t- and u-channels are scattering diagrams and have $t < 0$, $u < 0$.

See : $t = (p_1 - p_3)^2 = 2m^2 - 2 p_1 \cdot p_3 = 2m^2 - 2E_1E_3 + 2|\vec{p}_1||\vec{p}_3| \cos\theta.$

$$= 2m^2 - 2\sqrt{m^2 + |\vec{p}_1|^2}\sqrt{m^2 + |\vec{p}_3|^2} + 2\cos\theta\, |\vec{p}_1||\vec{p}_3|$$

$$< 0$$

If we have an interaction with derivatives in it,

$$\mathcal{L}_I = \lambda \phi_1 (\partial_\mu \phi_2)(\partial^\mu \phi_3)$$

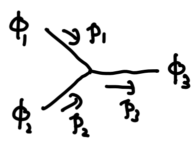
ϕ_1, ϕ_2 and ϕ_3 : three different scalars. In the momentum space, ∂_μ gives momenta

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x} \right)$$

Then

$$a_p^\dagger \rightarrow i p_\mu, \qquad a_p \rightarrow -i p_\mu$$

We consider $\phi_1 \phi_2 \rightarrow \phi_3$ processes :



Then $\langle 0 | T\{ \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3)\ (i\lambda) \int d^4y\ \phi_1(y) (\partial_\mu \phi_2(y)) (\partial^\mu \phi_3(y)) \} | 0 \rangle$

$$= (i\lambda) \int d^4y\ D_F(x_1 - y) [\partial_\mu D_F(x_2 - y)] [\partial^\mu D_F(x_3 - y)]$$

The T - matrix is

$$\langle p_3 | iT | p_1 p_2 \rangle = (i\lambda) \int d^4y\ [-i \int d^4x_1\, e^{-ip_1\cdot x_1} (\partial_1^2)] D_F(x_1 - y)$$

$$\times [-i \int d^4x_2\, e^{-ip_2\cdot x_2} (\partial_2^2)] [\partial_\mu D_F(x_2 - y)]$$

$$\times [-i \int d^4x_3\, e^{ip_3\cdot x} (\partial_3^2)] [\partial^\mu D_F(x_3 - y)]$$

$$\mathrm{I} = \left[-i\int d^4x_1\, e^{-ip_1\cdot x_1}(\partial_1^2)\right]\int \frac{d^4k_1}{(2\pi)^4}\,\frac{i}{k_1^2+i\varepsilon}\,e^{ik_1\cdot(x_1-y)}$$

$$= \int \frac{d^4k_1}{(2\pi)^4}\left[-i\int d^4x_1\, e^{-ip_1\cdot x_1}(ik_1)^2\right]\frac{i}{k_1^2+i\varepsilon}\,e^{ik_1\cdot(x_1-y)}$$

$$= -\int \frac{d^4k_1}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}(p_1-k_1)\,e^{-ik_1\cdot y}$$

$$= -\,e^{-ip_1\cdot y}$$

$$\mathrm{II} = \left[-i\int d^4x_2\, e^{-ip_2\cdot x_2}(\partial_2^2)\right]\int \frac{d^4k_2}{(2\pi)^4}\,\frac{i}{k_2^2+i\varepsilon}\,e^{ik_2\cdot(x_2-y)}(-ik_{2\mu})$$

$$= \int \frac{d^4k_2}{(2\pi)^4}\left[-i\int d^4x_2\, e^{-ip_2\cdot x_2}(ik_2)^2\,\frac{i}{k_2^2+i\varepsilon}\,e^{ik_2\cdot(x_2-y)}(-ik_{2\mu})\right]$$

$$= -\int \frac{d^4k_2}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}(p_2-k_2)\,e^{-ik_2\cdot y}\,(-ik_{2\mu})$$

$$= -\,e^{-ip_2\cdot y}\,(-ip_{2\mu})$$

$$\mathrm{III} = \left[-i\int d^4x_3\, e^{ip_3\cdot x_3}(\partial_3)^2\right]\int \frac{d^4k_3}{(2\pi)^4}\,\frac{i}{k_3^2+i\varepsilon}\,e^{ik_3\cdot(x_3-y)}\,(-ik_3^{\mu})$$

$$= -\,e^{ip_3\cdot y}\,(ip_3^{\mu})$$

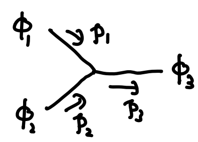
Then the T-matrix is

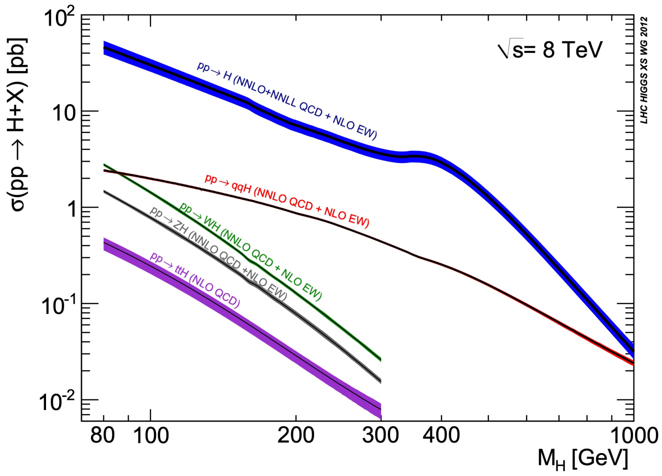
$$\langle p_3|\,iT\,|p_1 p_2\rangle = -(i\lambda)\int d^4y\; e^{iy\cdot(p_3-p_1-p_2)}\,(-ip_{2\mu})(ip_3^{\mu})$$

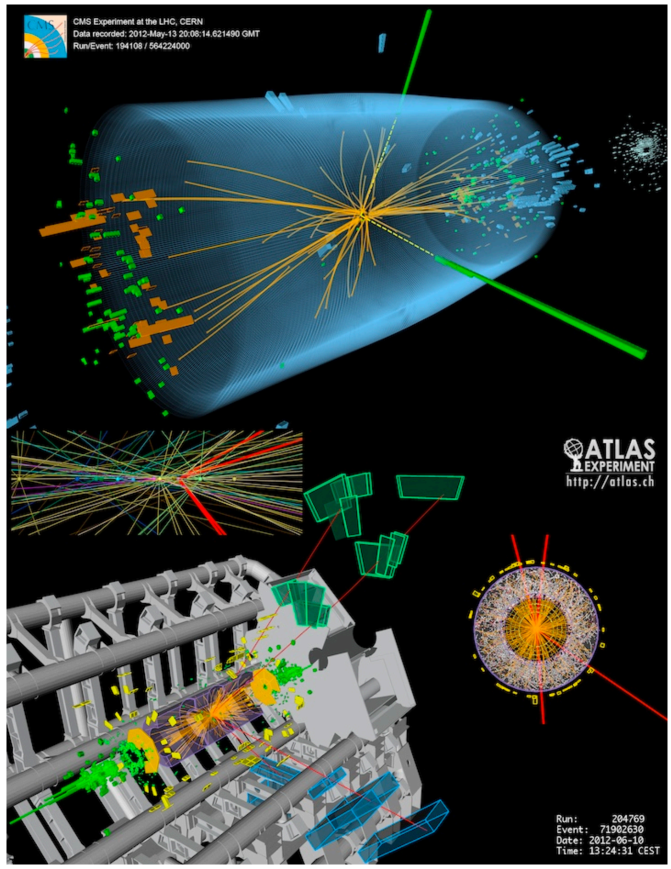
$$= -(i\lambda)\,(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3)\,(-ip_{2\mu})(ip_3^{\mu})$$

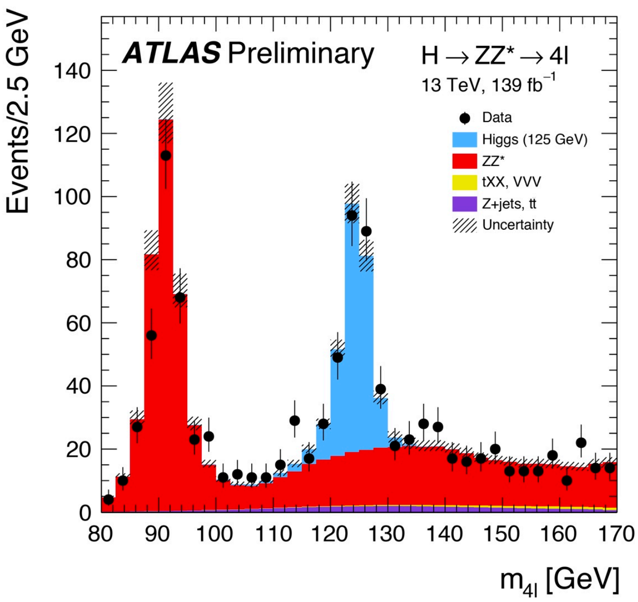
So

$$i\mathcal{M} = (-i\lambda)\,(-ip_{2\mu})(ip_3^{\mu})$$

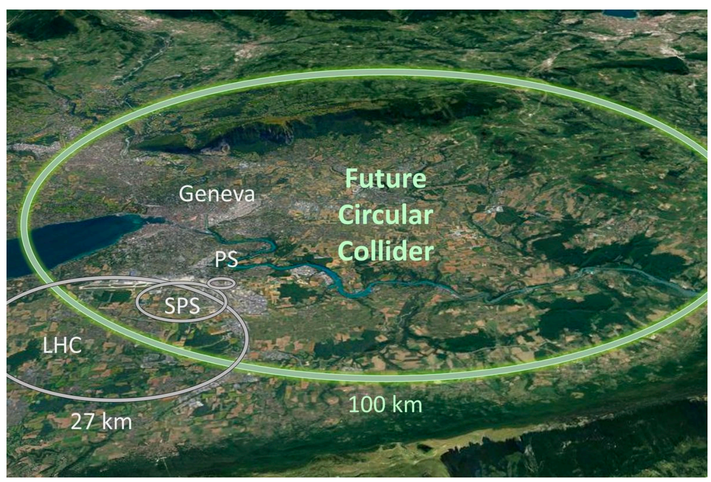



σ(pp → H+X) [pb]
√s= 8 TeV
LHC HIGGS XS WG 2012
pp → H (NNLO+NNLL QCD + NLO EW)
pp → qqH (NNLO QCD + NLO EW)
pp → WH (NNLO QCD + NLO EW)
pp → ZH (NNLO QCD +NLO EW)
pp → ttH (NLO QCD)
M_H [GeV]


CMS Experiment at the LHC, CERN
Data recorded: 2012-May-13 20:08:14.621490 GMT
Run/Event: 194108 / 564224000
ATLAS EXPERIMENT
http://atlas.ch
Run: 204769
Event: 71902630
Date: 2012-06-10
Time: 13:24:31 CEST


ATLAS Preliminary
H → ZZ* → 4l
13 TeV, 139 fb⁻¹
Data
Higgs (125 GeV)
ZZ*
tXX, VVV
Z+jets, tt
Uncertainty
Events/2.5 GeV
m_4l [GeV]


CMS
LHCb
ALICE
ATLAS


Geneva
Future
Circular
Collider
PS
SPS
LHC
27 km
100 km