4. Spin 1 fields.

- The group of translations and Lorentz transformations is called Poincaré group.
- Particles transform under irreducible unitary representations of the Poincaré group.
- There are no finite-dimensional unitary representation of the Poincaré group.
- Winger '93 showed that the irreducible unitary representations are uniquely classified by mass m and spin J. ($m \geq 0$, $J = 0, \frac{1}{2}, 1, \cdots$)
    - if $J > 0$, there are $2J+1$ independ states if $m > 0$

      . . . . 2 . . . . . . if $m = 0$.
    - if $J = 0$, there is only one state for any m.

4.1 Massive spin 1. (e.g. $W^{\pm}$, Z. Electroweak gauge boson) [Weinbeg 7.5 Schwartz 8.2.2]

For massive spin 1, there are three d.o.f.

We try to embed three d.o.f. in a vector field $A^{\mu}(x)$.

$$\underbrace{4}_{\substack{\text{4-dim representation}\\ \text{of the Lorentz group}}} = \underbrace{3}_{\substack{\text{3-dim representation}\\ \text{of } SO(3)\\ \text{"Spin - 1"}}} \oplus \underbrace{1}_{\substack{\text{1-dim representation}\\ \text{of } SO(3)\\ \text{"Spin - 0"}}}$$

The most general free Lagrangian is

$$\mathcal{L} = \frac{a}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{b}{2}(\partial_\mu A_\mu)(\partial^\nu A^\nu) + \frac{1}{2}m^2 A_\mu A^\mu$$

The E.-L. eq is

$$a\,\partial^2 A^\nu + b\,\partial^\nu \partial_\mu A^\mu - m^2 A^\nu = 0.$$

$$\boxed{\begin{aligned} &\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = a(\partial^\mu A^\nu) + b(\partial_\rho A^\rho) g^{\mu\nu} \quad ; \quad \frac{\partial \mathcal{L}}{\partial A^\mu} = m^2 A^\mu \\ &\partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right] = a\,\partial^2 A^\nu + b\,\partial^\nu(\partial_\rho A^\rho) \end{aligned}}$$

Taking ∂_ν on this equation

$$\Rightarrow a\,\partial^2 \partial_\nu A^\nu + b\,\partial^2 \partial_\mu A^\mu - m^2 \partial_\nu A^\nu = 0.$$

$$\Rightarrow [(a+b)\partial^2 - m^2](\partial_\mu A^\mu) = 0$$

If $a = -b$, and $m > 0$, then $\partial_\mu A^\mu = 0$

It is Lorentz invariant

It remove spin-0 representation.

If we choose $a = -1$, $b = 1$, then

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \qquad \text{[Proca Lagrangian]}$$

$\mathcal{L}$ is independent on $\dot{A}^0$, then $\pi^0 = 0$. A^0 is not a physical d.o.f.

Then the equation of motion now imply

$$(\partial^2 + m^2) A^\mu = 0 , \quad \partial_\mu A^\mu = 0 .$$

not gauge, remove one d.o.f.

$$A^0 = -\frac{1}{m^2} \nabla \cdot \vec{\pi}$$

$$\partial_\nu (\partial^\nu A^\mu - \partial^\mu A^\nu) + m^2 A^\mu = 0$$

$$\partial_\nu F^{\nu\mu} = -m^2 A^\mu$$

$$\partial_i F^{i0} = -m^2 A^0$$

$$\partial_i \pi^i = -m^2 A^0$$

$$A^0 = -\nabla \cdot \vec{\pi} / m^2$$

The energy momentum tensor is

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L}$$

$$= (-\partial^\mu A^\rho + \partial_\sigma A^\sigma g^{\mu\rho}) \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L}$$

$$= -\partial^\mu A^\rho \partial^\nu A_\rho + \partial_\sigma A^\sigma \partial^\nu A^\mu - g^{\mu\nu} \mathcal{L}$$

$$= -A^\sigma \partial^\nu \partial_\sigma A^\mu = \partial^\nu A^\sigma \partial_\sigma A^\mu$$

$$= -F^{\mu\rho} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L} .$$

Then T^{00} is

$$T^{00} = -F^{0\rho} \partial^0 A_\rho - \mathcal{L}$$

$$= -F^{0i} \partial^0 A_i + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu$$

$$F^{\mu\nu} F_{\mu\nu} = 2 F^{0i} F_{0i} + F^{ij} F_{ij} \quad \leftarrow \quad E^i = -F^{0i}$$

$$= 2 E^i E_i + \varepsilon^{ijk} B^k \varepsilon_{ij}{}^{l} B^l \qquad \varepsilon^{ijk} B^k = -F^{ij}$$

$$= -2 |\vec{E}|^2 + 2 |\vec{B}|^2$$

$$-F^{0i} \partial^0 A_i = -F^{0i} (\partial^0 A_i - \partial_i A^0 + \partial_i A^0)$$

$$= -F^{0i} F^0{}_i - F^{0i} \partial_i A^0$$

$$= E^2 - \partial_i (F^{0i} A^0) + (\partial_i F^{0i}) A^0$$

doesn't contribute to the total energy

$$= E^2 + (\partial_i \partial^0 A^i - \partial_i \partial^i A^0) A^0$$

$$= E^2 + \partial_0 (\partial_i A^i) A^0 + A^0 \partial^i \partial^i A^0$$

$$= E^2 + \partial_0 (\partial_0 A^0 + \partial_i A^i) A^0 - A^0 \partial_0 \partial_0 A^0 + A^0 \partial^i \partial^i A^0$$

$$= E^2 + A^0 \partial_0 (\partial_\mu A^\mu) - A^0 \partial^2 A^0$$

Then $$T^{00} = \frac{1}{2} E^2 + \frac{1}{2} B^2 + A^0 \partial_0 (\underbrace{\partial_\mu A^\mu}_{\partial_\mu A^\mu = 0.}) - A^0 \partial^2 A^0 - \frac{1}{2} m^2 A_\mu A^\mu$$

$$= \frac{1}{2} E^2 + \frac{1}{2} B^2 - A^0 \partial^2 A^0 - \frac{1}{2} m^2 (A^0)^2 + \frac{1}{2} m^2 A^2$$

$$= \frac{1}{2} E^2 + \frac{1}{2} B^2 + \frac{1}{2} m^2 A^2 + \frac{1}{2} m^2 (A^0)^2 - \underbrace{A^0 (\partial^2 + m^2) A^0}_{= 0 \quad \text{since } (\partial^2 + m^2) A^0 = 0}$$

Finally, we find the total energy is positive.

Now, we want to find the solution to $(\partial^2 + m^2) A^\mu = 0$, and $\partial^\mu A_\mu = 0$

If we write

$$A^\mu(x) = \sum_{\lambda=1}^{3} \int \frac{d^4 p}{(2\pi)^4} \, \varepsilon^\mu(\vec{p}, \lambda) \, e^{i p \cdot x}$$

Then $\partial^\mu A_\mu = 0$ implies $p_\mu \varepsilon^\mu(\vec{p}, \lambda) = 0$

For any fixed p_μ with $p^2 = m^2$, there are three independent solutions

Polarization vectors $\varepsilon^\mu(\vec{p}, \lambda)$, $\lambda = 1, 2, 3$, normalized by $\varepsilon^*_\mu \varepsilon^\mu = -1$

Taking p^μ in the rest frame, $p^\mu = (m, 0, 0, 0)$

Then $$\begin{cases} \varepsilon^\mu(p, 1) = (0, 1, 0, 0) \; ; \; \varepsilon^\mu(p, 2) = (0, 0, 1, 0) \\ \varepsilon^\mu(p, 3) = (0, 0, 0, 1) \end{cases}$$

Taking p^μ along z-direction $p^\mu = (E, 0, 0, p_3)$

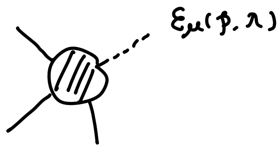
Then $\varepsilon_1^\mu = (0, 1, 0, 0)$, $\varepsilon_2^\mu = (0, 0, 1, 0)$, Transverse polarization

$\varepsilon_L^\mu = (p_3, 0, 0, E)/m$ longitudinal polarization

$$\boxed{\begin{aligned} &\varepsilon^\mu(p, \lambda) \, \varepsilon^*_\mu(p, \lambda') = -\delta^{\lambda\lambda'} = g^{\lambda\lambda'} \quad [\lambda, \lambda' = 1, 2, 3] \\ &\sum_{\lambda=1}^{3} \varepsilon_\mu(p, \lambda) \, \varepsilon^*_\nu(p, \lambda) = -g_{\mu\nu} + p_\mu p_\nu / m^2 \end{aligned}}$$

In the high energy limit $E \gg m$, $\varepsilon^\mu(p,3) \to E/m\,(1,0,0,1)$

In the scattering process with A^μ



$$d\sigma^T \sim g^2 \varepsilon_T^2 \sim g^2 \qquad [\,g : \text{coupling}\,]$$

$$d\sigma^L \sim g^2 \varepsilon_L^2 \sim g^2 E^2/m^2$$

↑ cross section blows up !

" cancelled by Higgs boson "

4.2 Quantization

To quantize fields with multiple d.o.f. , we need creation and annihilation operators for each d.o.f. seperately.

E.g. two scalar fields, ϕ_1 and ϕ_2

$$\phi_1(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{p,1} e^{-ip\cdot x} + a_{p,1}^{+} e^{ip\cdot x} \right)$$

$$\phi_2(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{p,2} e^{-ip\cdot x} + a_{p,1}^{+} e^{ip\cdot x} \right)$$

Then the complex field $\phi = \phi_1 + i\phi_2$

$$\phi(x) = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{j=1}^{2} \left(\vec{\varepsilon}_j a_{p,j} e^{-ip\cdot x} + \vec{\varepsilon}_j a_{p,j}^{+} e^{ip\cdot x} \right)$$

with $\vec{\varepsilon}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{\varepsilon}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ [polarization vectors]

Massive spin - 1

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{\lambda=1}^{3} \left[\varepsilon_\mu(p,\lambda) a_{p,\lambda} e^{-ip\cdot x} + \varepsilon_\mu^*(p,\lambda) a_{p,\lambda}^{+} e^{ip\cdot x} \right]$$

Hermitian

$$(\omega_p = \sqrt{m^2 + |\vec{p}|^2})$$

Here $a_{p,\lambda}$ and $a_{p,\lambda}^{+}$ have polarization indices

$$|\vec{p}, \lambda\rangle = \sqrt{2\omega_p}\, a_{p,\lambda}^{+} |0\rangle$$

$$\langle 0 | A_\mu(x) | \vec{p}, \lambda \rangle = \varepsilon_\mu(\vec{p}, \lambda) e^{-ip\cdot x}$$

Cannonical quantization

$$[a_{p,\lambda} , a_{p',\lambda'}^{+}] = \delta_{\lambda\lambda'} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')$$

$$[a_{p,\lambda} , a_{p',\lambda'}] = [a_{p,\lambda}^{+} , a_{p',\lambda'}^{+}] = 0$$

We first compute

$$[\, A_\mu(x),\ A_\nu(y)\,] = \sum_{\lambda,\lambda'} \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}}\, \frac{d^3q}{(2\pi)^3\sqrt{2\omega_q}}\, [\, \varepsilon_\mu(p,\lambda)\, a_{p,\lambda}\, e^{-ip\cdot x} + \varepsilon^*_\mu(p,\lambda)\, a^\dagger_{p,\lambda}\, e^{ip\cdot x},$$

$$\varepsilon_\nu(q,\lambda')\, a_{q,\lambda'}\, e^{-iq\cdot y} + \varepsilon^*_\nu(q,\lambda')\, a^\dagger_{q,\lambda'}\, e^{iq\cdot y}\,]$$

$$= \cdots\cdots \Big\{ [\, \varepsilon_\mu(p,\lambda)\, a_{p,\lambda}\, e^{-ip\cdot x},\ \varepsilon^*_\nu(q,\lambda')\, a^\dagger_{q,\lambda'}\, e^{iq\cdot y}\,]$$

$$+ [\, \varepsilon^*_\mu(p,\lambda)\, a^\dagger_{p,\lambda}\, e^{ip\cdot x},\ \varepsilon_\nu(q,\lambda')\, a_{q,\lambda'}\, e^{-iq\cdot y}\,] \Big\}$$

$$= \sum_\lambda \int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\, [\, e^{-ip\cdot x + ip\cdot y}\, \varepsilon_\mu(p,\lambda)\, \varepsilon^*_\nu(p,\lambda) - e^{ip\cdot(x-y)}\, \varepsilon^*_\mu(p,\lambda)\, \varepsilon_\nu(p,\lambda)\,]$$

insert $[\, a,\ a^\dagger\,] = (2\pi)^3\delta^{(3)}(p-q)\,\delta_{\lambda,\lambda'}$

$$= \int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\, [\, e^{-ip\cdot(x-y)}\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right) - e^{ip\cdot(x-y)}\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)]$$

$$= \left(-g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{m^2}\right) \Delta(x-y) = \Delta_{\mu\nu}(x-y)$$

As $(x-y)^2 < 0$, $[\, A_\mu(x),\ A_\nu(y)\,] = 0$.

Since $\pi^\mu = -F^{0\mu} \Rightarrow \pi^i = -F^{0i} = -\dot{A}^i + \partial^i A^0$

$$[\, A^i(t,\vec{x}),\ \pi^j(t,\vec{y})\,] = -[\, A^i(t,\vec{x}),\ \dot{A}^j(t,\vec{y})\,] + [\, A^i(t,\vec{x}),\ \partial^j A^0(t,\vec{y})\,]$$

$$= -\left[\frac{\partial}{\partial y^0}\Delta^{ij}(x-y)\right]_{y^0=x^0=t} + \partial^j \Delta^{i0}(x-y)\Big|_{x^0=y^0=t}$$

Here $\left[\frac{\partial}{\partial y^0}\Delta^{ij}(x-y)\right]_{y^0=x^0=t}$

$$= \int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\, [\, e^{-ip\cdot(x-y)}(ip^0) + (ip^0)\, e^{ip\cdot(x-y)}\,] \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)\Big|_{x^0=y^0=t}$$

$$= \left(\frac{i}{2}\right) \int \frac{d^3p}{(2\pi)^3}\, [\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\,] \left(-g^{ij} + \frac{p^i p^j}{m^2}\right)$$

$$= i\,\delta^{ij}\,\delta^{(3)}(\vec{x}-\vec{y}) + \left(\frac{i}{2}\right)\int \frac{d^3p}{(2\pi)^3}\, [\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\,]\, \frac{p^i p^j}{m^2}$$

$$\partial^j \Delta^{i0}(x-y)\Big|_{x^0=y^0=t}$$

$$= \int \frac{d^3p}{(2\pi)^3\, 2\omega_p}\, [\, e^{-ip\cdot(x-y)}(ip^j) + (ip^j)\, e^{ip\cdot(x-y)}\,] \left(-\underset{0}{\underset{\shortparallel}{g^{i0}}} + p^i p^0/m^2\right)\Big|_{x^0=y^0=t}$$

$$= \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\, [\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\,]\, p^i p^j/m^2$$

$$\Rightarrow [\, A^i(t,\vec{x}),\ \pi^j(t,\vec{y})\,] = -i\,\delta^{ij}\,\delta^{(3)}(\vec{x}-\vec{y})$$

Similarly, one has

$$[\, A^i(t,\vec{x}),\ A^j(t,\vec{y})\,] = [\, \pi^i(t,\vec{x}),\ \pi^j(t,\vec{y})\,] = 0.$$

They are not covariant. Therefore we can define the field operator

$$A^0(t,\vec{x}) = -\nabla\cdot\vec{\pi}\,/m^2$$

which corresponds the classical relation.

One can also evaluate the Hamiltonian and momentum operators

$$H = \sum_{\lambda=1}^{3}\int d^3p\ \ \omega_p\ a^{+}_{p,\lambda}\, a_{p,\lambda}$$

$$P^i = \sum_{\lambda=1}^{3}\int d^3p\ \ p^i\ a^{+}_{p,\lambda}\, a_{p,\lambda}$$

The spin operator

Under the Lorentz transformation. $A^\mu(x) \to \Lambda^\mu{}_\nu\, A^\nu(\Lambda^{-1}x)$

infinitesimal transformations of the Lorentz group.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \underset{\substack{\uparrow \\ \text{infinitesimal } \omega}}{\omega^\mu{}_\nu}$$

Then $\quad \Lambda^\mu{}_\sigma\, g^{\sigma\rho}\, \Lambda^\nu{}_\rho = g^{\mu\nu} \longrightarrow \omega^{\mu\nu} + \omega^{\nu\mu} = 0 \quad$ [antisymmetric]

An antisymmetric 4×4 matrix has 6 independent components :

6 transformations of the Lorentz group : 3 rotations & 3 boosts

One possible basis : $\quad (\mathcal{J}^A)^\mu{}_\nu \quad (A = 1, 2, \cdots, 6)$

In practice, replacing the single index A with $[\rho\sigma]$ $\quad (\rho,\sigma = 0,\cdots,3)$

antisymmetric $\quad$ e.g. $\mu^{01} = -\mu^{10}$

With this rotation, we can write a basis as

$$(\mathcal{J}^{\rho\sigma})^\mu{}_\nu = i\left(g^{\rho\mu}g^{\sigma\nu} - g^{\sigma\mu}g^{\rho\nu}\right)$$

$\begin{cases} \rho\sigma : & \text{index of basis elements} \\ \mu\nu : & \text{index of the } 4\times 4 \text{ matrix} \end{cases}$

Any $\omega^\mu{}_\nu$ can be expressed as a linear combination of $\mathcal{J}^{\rho\sigma}$

$$\omega^\mu{}_\nu = \frac{i}{2} \underbrace{\Omega_{\rho\sigma}}_{\text{6 numbers}} \underbrace{(\mathcal{J}^{\rho\sigma})^\mu{}_\nu}_{\text{the generators of the Lorentz transformation}}$$

The generators obey the Lorentz Lie algebra

$$[\mathcal{J}^{\rho\sigma}, \mathcal{J}^{\tau\nu}] = i\left(g^{\sigma\tau}\mathcal{M}^{\rho\nu} - g^{\rho\tau}\mathcal{M}^{\sigma\nu} + g^{\rho\nu}\mathcal{M}^{\sigma\tau} - g^{\sigma\nu}\mathcal{M}^{\rho\tau}\right)$$

Therefore, the Lorentz transformation on $A^\mu(x)$ is

$$\begin{aligned} A^\mu(x) \longrightarrow\ & \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) \\ &= (\delta^\mu{}_\nu + \omega^\mu{}_\nu) A^\nu(x^\rho - \omega^\rho{}_\sigma x^\sigma) \\ &= A^\mu(x^\rho - \omega^\rho{}_\sigma x^\sigma) + \omega^\mu{}_\nu A^\nu(x^\rho - \omega^\rho{}_\sigma x^\sigma) \\ &= A^\mu(x) \underbrace{- \omega^\rho{}_\sigma x^\sigma \partial_\rho A^\mu}_{\text{gives orbital angular momentum}} + \underbrace{\omega^\mu{}_\nu A^\nu(x)}_{\text{describes intrisic angular momentum (the spin) of the vector fields.}} + O(\omega^2) \end{aligned}$$

Then the corresponding conserved current is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \omega^\nu{}_\alpha A^\alpha(x) &= -F^\mu{}_\nu \frac{i}{2}\Omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\nu{}_\alpha A^\alpha(x) \\ &= -F^\mu{}_\nu \frac{i}{2}\Omega_{\rho\sigma} (i)(g^{\rho\nu}\delta^\sigma{}_\alpha - \delta^\rho{}_\alpha g^{\sigma\nu}) A^\alpha(x) \\ &= \frac{\Omega_{\rho\sigma}}{2}(F^{\mu\rho}A^\sigma - F^{\mu\sigma}A^\rho) \end{aligned}$$

$$\Rightarrow \quad (\mathcal{J}^\mu)^{\rho\sigma} = F^{\mu\rho}A^\sigma - F^{\mu\sigma}A^\rho$$

For $\rho, \sigma = 1, 2, 3$

$$Q^{ij} = \int d^3x\, (F^{0j}A^i - F^{0i}A^j)$$

Writing in terms of three vectors,

$$\begin{aligned} S^k = \frac{1}{2}\varepsilon^{ijk} Q^{ij} &= \frac{1}{2}\varepsilon^{ijk} \int d^3x\, (F^{0j}A^i - F^{0i}A^j) \\ &= \frac{1}{2}\int d^3x\, (-\varepsilon^{ijk}\pi^j A^i + \varepsilon^{ijk}\pi^i A^j) \\ \vec{S} &= \int d^3x\, \vec{\pi} \times \vec{A} \end{aligned}$$

Then the spin operator

$$S = \int d^3x \; N(\vec{\pi} \times \vec{A})$$
$\uparrow$
normal order

The conjugate momentum

$$\pi^i = -\dot{A}^i + \partial^i A^0$$
$$= i\int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{\lambda=1}^{3} \omega_k \tilde{\varepsilon}^i(k,\lambda) \left(a_{k,\lambda} e^{-ik\cdot x} - a^+_{k,\lambda} e^{ik\cdot x} \right)$$

with $\tilde{\varepsilon}^i(k,\lambda) \equiv \varepsilon^i(k,\lambda) - k^i \varepsilon^0(k,\lambda)/\omega_k$

$$= \varepsilon^i - \frac{\vec{k}\cdot\vec{\varepsilon}}{\omega^2} k^i \qquad [k^0\varepsilon^0 - \vec{k}\cdot\vec{\varepsilon} = 0]$$

also $\quad \vec{k}\cdot\vec{\tilde{\varepsilon}}(k,\lambda) = \vec{k}\cdot\vec{\varepsilon}\left(1 - \frac{\vec{k}^2}{\omega_k^2}\right) = \vec{k}\cdot\vec{\varepsilon}\,\frac{m^2}{\omega_k^2}$

So $\quad \tilde{\varepsilon}(k,\lambda) = f_\lambda \varepsilon(k,\lambda) \quad$ with $f_\lambda = \begin{cases} 1, & \lambda = 1,2 \\ m^2/\omega_k^2, & \lambda = 3 \end{cases}$

Then, $\vec{S} = \int d^3x \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} \frac{d^3q}{(2\pi)^3\sqrt{2\omega_q}} \sum_{\lambda,\lambda'} (i\omega_q)\, \vec{\tilde{\varepsilon}}(q,\lambda') \times \vec{\varepsilon}(p,\lambda)$

$$\times N\left[\left(a_{q,\lambda'} e^{-iq\cdot x} - a^+_{q,\lambda'} e^{iq\cdot x} \right)\left(a_{p,\lambda} e^{-ip\cdot x} + a^+_{p,\lambda} e^{ip\cdot x} \right) \right]$$

$$= \sum_{\lambda,\lambda'} \int \frac{d^3p}{(2\pi)^3} \left(\frac{i}{2}\right) \Big[\vec{\tilde{\varepsilon}}(-p,\lambda') \times \vec{\varepsilon}(p,\lambda) \left(a_{-p,\lambda'} a_{p,\lambda} e^{-i2\omega_p t} - a^+_{-p,\lambda'} a^+_{p,\lambda} e^{i2\omega_p t} \right)$$
$$+ \vec{\tilde{\varepsilon}}(p,\lambda') \times \vec{\varepsilon}(p,\lambda) \left(a^+_{p,\lambda} a_{p,\lambda'} - a^+_{p,\lambda'} a_{p,\lambda} \right) \Big]$$

define helicity operator $h = \vec{S}\cdot\vec{p}/|\vec{p}|$, projection of the spin in the direction of motion.

as $\lambda' = 3$, $[\vec{\tilde{\varepsilon}}(-p,3) \times \vec{\varepsilon}(p,\lambda)]\cdot\vec{p} = [\vec{p} \times \vec{\tilde{\varepsilon}}(-p,3)]\cdot\vec{\varepsilon}(p,\lambda)$

$$= \frac{m^2}{\omega_p^2}[\vec{p} \times \vec{\varepsilon}(-p,3)]\cdot\vec{\varepsilon}(p,\lambda) = 0.$$

$\lambda = 3$, $[\vec{\tilde{\varepsilon}}(-p,\lambda') \times \vec{\varepsilon}(p,3)]\cdot\vec{p} = 0$

For the transverse polarizations, $\lambda = 1, 2$, $\tilde{\varepsilon}(p,\lambda) = \varepsilon(p,\lambda)$

vanish after $\lambda \leftrightarrow \lambda'$, $p \leftrightarrow -p$

$$h = \sum_{\lambda,\lambda'} \int \frac{d^3p}{(2\pi)^3} \left(\frac{i}{2}\right) \Big[\vec{\varepsilon}(-p,\lambda') \times \vec{\varepsilon}(p,\lambda) \left(a_{-p,\lambda'} a_{p,\lambda} e^{-i2\omega_p t} - a^+_{-p,\lambda'} a^+_{p,\lambda} e^{i2\omega_p t} \right.$$

$$\left. + \vec{\varepsilon}(p,\lambda') \times \vec{\varepsilon}(p,\lambda) \left(a^+_{p,\lambda} a_{p,\lambda'} - a^+_{p,\lambda'} a_{p,\lambda} \right) \Big] \cdot \frac{\vec{p}}{|\vec{p}|}$$

$$= \sum_{\lambda,\lambda'=1}^{2} \int \frac{d^3p}{(2\pi)^3} \left(\frac{i}{2}\right) \vec{e}_p \cdot \left[\vec{\varepsilon}(p,\lambda') \times \vec{\varepsilon}(p,\lambda) \right] \left(a^+_{p,\lambda} a_{p,\lambda'} - a^+_{p,\lambda'} a_{p,\lambda} \right) \qquad \left[\vec{e}_p = \vec{p}/|\vec{p}| \right]$$

$$\vec{e}_p \cdot \left[\vec{\varepsilon}(p,1) \times \vec{\varepsilon}(p,2) \right] = 1$$

$$= i \int \frac{d^3p}{(2\pi)^3} \left(a^+_{p,2} a_{p,1} - a^+_{p,1} a_{p,2} \right)$$

Thus states of $a^+_{p,\lambda}|0\rangle$ are not eigenstates of the helicity operator.

This can be achieved by going to the helicity basis.

define
$$\begin{cases} a_{p,+} = \frac{1}{\sqrt{2}} (a_{p,1} - i a_{p,2}) \\ a_{p,-} = \frac{1}{\sqrt{2}} (a_{p,1} + i a_{p,2}) \\ a_{p,0} = a_{p,3} \end{cases}$$

The commutators are still $[a_{p',\pm 0} , a^+_{p,\pm 0}] = (2\pi)^3 \delta^{(3)}(\vec{p}' - \vec{p})$

Then the helicity operator

$$\hat{h} = \int \frac{d^3p}{(2\pi)^3} \left(a^+_{p,+} a_{p,+} - a^+_{p,-} a_{p,-} \right)$$

The helicity polarization vectors are

$$\begin{cases} \varepsilon^\mu(k,\pm) = \frac{1}{\sqrt{2}} \left[\varepsilon^\mu(k,1) \pm i\, \varepsilon^\mu(k,2) \right] \\ \varepsilon^\mu(k,0) = \varepsilon^\mu(k,3) \end{cases}$$

Feynman propagator : $D_F^{\mu\nu}(x-y)$

$$D_F^{\mu\nu}(x-y) = \langle 0| T\{ A^\mu(x) A^\nu(y) \} |0\rangle$$

$$= \theta(x^0-y^0)\left(-g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right) D(x-y) + \theta(y^0-x^0)\left(-g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}\right) D(y-x)$$

Since $\quad \partial_0 [\theta(x^0-y^0) D(x-y)] = \delta(x^0-y^0) D(x-y) + \theta(x^0-y^0)\partial_0 D(x-y)$

$$\partial_0^2 [\theta(x^0-y^0) D(x-y)] = \delta'(x^0-y^0) D(x-y) + 2\delta(x^0-y^0)\partial_0 D(x-y) + \theta(x^0-y^0)\partial_0^2 D(x-y)$$

$$\partial_0^2 [\theta(y^0-x^0) D(y-x)] = -\delta'(x^0-y^0) D(y-x) - 2\delta(x^0-y^0)\partial_0 D(y-x) + \theta(y^0-x^0)\partial_0^2 D(y-x)$$

Then we have

$$\theta(x^0-y^0)\partial_0^2 D(x-y) + \theta(y^0-x^0)\partial_0^2 D(y-x)$$

$$= \partial_0^2 D_F(x-y) - \delta'(x^0-y^0)[\phi(x),\phi(y)] - 2\delta(x^0-y^0)\frac{\partial}{\partial x^0}[\phi(x),\phi(y)]$$

where $\quad \delta(x^0-y^0)[\underset{\pi(x)}{\underset{\uparrow}{\dot\phi(x)}},\phi(y)] = -i\delta^{(3)}(\vec{x}-\vec{y})$

$$\delta'(x^0-y^0)[\phi(x),\phi(y)] = \frac{\partial}{\partial x^0}[\underbrace{\delta(x^0-y^0)[\phi(x),\phi(y)]}_{=0}] - \delta(x^0-y^0)[\dot\phi(x),\phi(y)]$$

$$= i\delta^{(3)}(\vec{x}-\vec{y})$$

So $\quad \theta(x^0-y^0)\partial_0^2 D(x-y) + \theta(y^0-x^0)\partial_0^2 D(y-x)$

$$= \partial_0^2 D_F(x-y) + i\delta^{(3)}(\vec{x}-\vec{y})\delta(x^0-y^0)$$

$$= \partial_0^2 D_F(x-y) + i\delta^{(4)}(x-y)$$

Finally. $$\boxed{D_F^{\mu\nu}(x-y) = (-g^{\mu\nu} - \partial^\mu\partial^\nu/m^2) D_F(x-y) - \frac{i}{m^2} g^{\mu 0} g^{\nu 0}\delta^{(4)}(x-y)}$$

don't contribute to physical results

See : $\quad \theta(x^0-y^0)\partial_0 D(x-y) + \theta(y^0-x^0)\partial_0 D(y-x)$

$$= \partial_0[\theta(x^0-y^0) D(x-y) + \theta(y^0-x^0) D(y-x)] \underbrace{- \delta(x^0-y^0) D(x-y) + \delta(x^0-y^0) D(y-x)}_{= -\delta(x^0-y^0)[\phi(x),\phi(y)] = 0}.$$

Transition to interaction picture

e.g. a neutral scalar field with derivative coupling

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - j^\mu\partial_\mu\phi$$

external current or a functional of other fields

Then canonical momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi - j^0$$

The Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \pi\dot\phi - \mathcal{L}\\ &= \pi\dot\phi - \left(\tfrac{1}{2}\dot\phi^2 - \tfrac{1}{2}(\nabla\phi)^2 - j^0\dot\phi - \vec{j}\cdot\nabla\phi\right)\end{aligned}$$

$\Big\downarrow$ replace $\dot\phi$ by $\pi + j^0$

$$\begin{aligned}&= \pi(\pi+j^0) - \tfrac{1}{2}(\pi+j^0)^2 + \tfrac{1}{2}(\nabla\phi)^2 + j^0(\pi+j^0) + \vec{j}\cdot\nabla\phi\\ &= \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)^2 \qquad = \mathcal{H}_0\\ &\quad\ \pi j^0 + \tfrac{1}{2}(j^0)^2 + \vec{j}\cdot\nabla\phi \qquad = V\end{aligned}$$

Here $\mathcal{H}_0$ is free Hamiltonian

V is the interaction

The last step is replacing V with V_I (I-picture, in terms of free fields)

$$V_I = \dot\phi\, j^0 + \vec{j}\cdot\nabla\phi + \tfrac{1}{2}(j^0)^2 = j^\mu\partial_\mu\phi + \tfrac{1}{2}(j^0)^2$$

replace π with $\dot\phi$

cancel a non-invariant term in the propagator (see later)

Let's consider the Lagrangian of a massive vector A^μ interacting with a current j_μ:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - j_\mu A^\mu$$

Then the E.-L. equation is

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu$$

$$\nu = 0 \quad : \quad \partial_i F^{i0} + m^2 A^0 = j^0 \qquad [\pi^i = F^{i0} = E^i, \ \pi^0 = 0]$$

$$\Rightarrow \quad A^0 = (-\nabla\cdot\vec{\pi} + j^0)/m^2 .$$

Since $\quad \vec{\pi} = -\dot{\vec{A}} - \nabla A^0 = -\dot{\vec{A}} + \nabla(\nabla\cdot\vec{\pi} - j^0)/m^2$

$$\Rightarrow \quad \dot{\vec{A}} = \nabla(\nabla\cdot\vec{\pi} - j^0)/m^2 - \vec{\pi}$$

Then the Hamiltonian is

$$\mathcal{H} = \pi^\mu \dot{A}_\mu - \mathcal{L} = -\vec{\pi}\cdot\dot{\vec{A}} - \mathcal{L} \equiv H_0 + V$$

$$= -\vec{\pi}\cdot\left[\nabla(\nabla\cdot\vec{\pi} - j^0)/m^2 - \vec{\pi}\right] + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} m^2 A_\mu^2 + j^\mu A_\mu .$$

$$= \vec{\pi}^2 + \vec{\pi}\cdot\nabla j^0/m^2 - \vec{\pi}\cdot\nabla(\nabla\cdot\vec{\pi})/m^2 + \frac{1}{2} F_{0i}F^{0i} + \frac{1}{4} F_{ij}F^{ij} - \frac{1}{2} m^2 (A_0^2 - \vec{A}^2)$$

$$+ j^0 A^0 - \vec{j}\cdot\vec{A}$$

$$\vec{\pi}\cdot\nabla j^0/m^2 = -j^0\,\nabla\cdot\vec{\pi}/m^2$$

$$-\vec{\pi}\cdot\nabla(\nabla\cdot\vec{\pi})/m^2 = +(\nabla\cdot\vec{\pi})^2/m^2$$

$$\frac{1}{2} F_{0i}F^{0i} = -\frac{1}{2}\vec{\pi}^2$$

$$\frac{1}{4} F_{ij}F^{ij} = \frac{1}{2}(\nabla\times\vec{A})^2 \qquad [\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}]$$

$$-\frac{1}{2} m^2 A^{0\,2} = -\frac{1}{2} m^2 (\nabla\cdot\vec{\pi} - j^0)^2/m^4 = -\frac{1}{2m^2}\left[(\nabla\cdot\vec{\pi})^2 - 2 j^0\,\nabla\cdot\vec{\pi} + j^{0\,2}\right]$$

$$j^0 A^0 = -j^0(\nabla\cdot\vec{\pi} - j^0)/m^2 = -j^0(\nabla\cdot\vec{\pi})/m^2 + (j^0)^2/m^2$$

$$= \frac{1}{2}\vec{\pi}^2 + \frac{1}{2}(\nabla\times\vec{A})^2 + \frac{1}{2m^2}(\nabla\cdot\vec{\pi})^2 + \frac{1}{2} m^2 |\vec{A}|^2 \qquad \equiv H_0$$

$$-\vec{j}\cdot\vec{A} - \frac{1}{m^2} j^0(\nabla\cdot\vec{\pi}) + \frac{1}{2m^2} j^{0\,2} \qquad \equiv V$$

Then use $\nabla \cdot \vec{\pi} = -m^2 A^0$ to get V_I

$$V_I = -\vec{j} \cdot \vec{A} + j^0 A^0 + \frac{1}{2m^2} j^{0\,2}$$

$$= j^\mu A_\mu + \frac{1}{2m^2} (j^0)^2$$

Let's consider the following correlation function

$$\langle 0 | T \{ \exp[-i \int dt\, V_I(t)] \} | 0 \rangle$$

$$= 1 + \frac{(-i)^2}{2} \int d^4x\, d^4y \;\; j^\mu(x)\, j^\nu(y)\, D_{\mu\nu}(x-y)$$

$$- \frac{i}{2m^2} \int d^4x \; [j^0(x)]^2 \;\; + \cdots$$

The Feynman propagator is

$$\boxed{D_F^{\mu\nu}(x-y) = (-g^{\mu\nu} - \partial^\mu \partial^\nu / m^2)\, D_F(x-y) - \frac{i}{m^2} g^{\mu 0} g^{\nu 0} \delta^{(4)}(x-y)}$$

The contribution from the last term is

$$-\frac{1}{2} \int d^4x\, d^4y \;\; j_\mu(x)\, j_\nu(y) \;\; (-\frac{i}{m^2})\, g^{\mu 0} g^{\nu 0} \delta^{(4)}(x-y)$$

$$= \frac{i}{2m^2} \int d^4x\, [j^0(x)]^2$$

which cancel the contribution from $\frac{(j^0)^2}{2m^2}$ in V_I.

Constraints and Dirac Brackets

[Weinberg 7.6]

Proca Lagrangian (real massive vector field)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - j_\mu A^\mu$$

Primary constrains : imposed on the system (e.g. choose a gauge)

or arise from the Lagrangian itself.

e.g. $\mathcal{L}$ is independent on $\dot{A}^0$, then $\pi^0 = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0$

so the primary constraint $\pi^0 = 0$

Secondary constraints : primary constraints are consistent with the equation

of motion (e.o.m.)

e.g. the E.-L. equation

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu \quad \longrightarrow \quad \partial_i F^{i0} + m^2 A^0 = j^0 \quad (\nu = 0)$$

$$\partial_i \underset{\substack{\uparrow \\ \pi^i = F^{i0}}}{\pi^i} + m^2 A^0 = j^0$$

$\partial_i \pi^i + m^2 A^0 = j^0$ is consistent with $\pi^0 = 0$

Here we are finished, but in other theories we might encouter further constraints

by requring consistency of the sencondary constraints with E.-L. equation

Note :

- primary , secondary , ... ; the distinction is not important

first and second constraints

π_a, ψ^a will in general not be independent variables, but be related by constraints

Usually, they fulfill

$$\{\psi^a, \pi_b\}_p = \delta^a{}_b \quad ; \quad \{\psi^a, \psi^a\}_p = \{\pi_a, \pi_a\}_p = 0$$

Poisson bracket : $\{A, B\}_p = \frac{\delta A}{\delta \psi^a}\frac{\delta B}{\delta \pi_a} - \frac{\delta A}{\delta \pi_a}\frac{\delta B}{\delta \psi^a}$ (a includes position)

Canonical quantization :

$$\{\ ,\ \}_p \longrightarrow \frac{1}{i}[\ ,\ ]$$

But the constraints do not always allow this

Express constraints as $\chi_N(\{\psi^a\}, \{\pi_a\}) = 0$ (N including position $\vec{x}$)

consistent with e.o.m. $\dot{\chi}_N = \{\chi_N, H\}_p = 0$

First class constraints :

$$\{\chi_N, \chi_M\}_p = 0$$

χ_N's Poisson braket vanishes with all the other constraints

Note :

- arises from gauge invariance. (see later)
- associates with a symmetry

Second class constraints : the remaining constraints

define $C_{NM} \equiv \{\chi_N, \chi_M\}_p$ $\det C \neq 0$

e.g. in Proca theory, the constraints are 2nd class constraints

$$\chi_{1\vec{x}} = \pi^0(t, \vec{x}) = 0$$

$$\chi_{2\vec{x}} = \partial_i \pi^i(t, \vec{x}) + m^2 A^0(t, \vec{x}) - j^0(t, \vec{x}) = 0$$

Then $C = \begin{pmatrix} C_{1x,1y} & C_{1x,2y} \\ C_{2x,1y} & C_{2x,2y} \end{pmatrix} = \begin{pmatrix} 0 & -m^2\delta^{(3)}(x-y) \\ m^2\delta^{(3)}(x-y) & 0 \end{pmatrix}$, $\det C \neq 0$

See :

$$C_{1x,2y} = \{ \chi_{1x}, \chi_{2y} \}_P$$

$$= \int d^3z \left[\frac{\delta \chi_{1x}}{\delta A^\mu(t,\vec{z})} \frac{\delta \chi_{2y}}{\delta \pi_\mu(t,\vec{z})} - \frac{\delta \chi_{1x}}{\delta \pi_\mu(t,\vec{z})} \frac{\delta \chi_{2y}}{\delta A^\mu(t,\vec{z})} \right]$$

$$= \int d^3z \left[0 - \delta^{(3)}(\vec{x}-\vec{z})\, m^2 \delta^{(3)}(\vec{y}-\vec{z}) \right]$$

$$= -m^2 \delta^{(3)}(\vec{x}-\vec{y})$$

Dirac braket

If all constraints are second classes, define Dirac braket

$$\{A, B\}_D \equiv \{A, B\}_P - \{A, \chi_N\}_P (C^{-1})^{NM} \{\chi_M, B\}_P$$

Canonical quantization:

$$\{\ ,\ \}_D \longrightarrow \frac{1}{i}[\ ,\ ]$$

Note :

- Dirac braket is equal to the Poisson braket calculated in terms of the reduced set of unconstrained canonical variables.

e.g. Proca theory

$$C^{-1} = \begin{pmatrix} 0 & , & m^{-2}\delta^{(3)}(x-y) \\ -m^{-2}\delta^{(3)}(x-y) & , & 0 \end{pmatrix}$$

$\longleftarrow \cdots$ $[\delta^{(3)}(x-y)]^{-1} = \delta^{(3)}(x-y)$

$\int dz\, \delta(x-z) f(z,y) = \delta(x-y)$

$\hookrightarrow$ $f(z,y) = \delta(z-y)$

then

$$\{A^\mu(t,\vec{x}),\ \pi_\nu(t,\vec{y})\}_D$$

$$= \{A^\mu(t,\vec{x}),\ \pi_\nu(t,\vec{y})\}_P$$

$$- \int d^3x'\, d^3y' \ \{A^\mu(t,\vec{x}),\ \chi_{1x'}\}\,(C^{-1})_{1x',2y'}\,\{\chi_{2y'},\ \pi_\nu(t,\vec{y})\}_P$$

$$- \int d^3x'\, d^3y' \ \{A^\mu(t,\vec{x}),\ \chi_{2x'}\}\,(C^{-1})_{2x',1y'}\,\{\chi_{1y'},\ \pi_\nu(t,\vec{y})\}_P$$

$$\{A^\mu(t,\vec{x}),\ \pi_\nu(t,\vec{y})\}_P = \delta^\mu{}_\nu\, \delta^{(3)}(\vec{x}-\vec{y})$$

$$\{A^\mu(t,\vec{x}),\ \chi_{1x'}\}_P = \{A^\mu(t,\vec{x}),\ \pi^0(t,x')\}_P = \delta^\mu{}_0\, \delta^{(3)}(x-x')$$

$$\{\chi_{2y'},\ \pi_\nu(t,\vec{y})\}_P = \{m^2 A^0(t,y'),\ \pi_\nu(t,\vec{y})\}_P = m^2\, \delta^0_\nu\, \delta^{(3)}(y-y')$$

$$\{\chi_{1y'},\ \pi_\nu(t,\vec{y})\}_P = \{\pi^0(t,\vec{y}'),\ \pi_\nu(t,\vec{y})\}_P = 0$$

$$= \delta^\mu{}_\nu\, \delta^{(3)}(x-y) - \delta^\mu{}_0\, \delta^0_\nu\, \delta^{(3)}(x-y)$$

$\hookrightarrow$ $$\{A^i(t,\vec{x}),\ \pi_j(t,\vec{y})\}_D = \delta^i{}_j\, \delta^{(3)}(x-y)$$

$\hookrightarrow$ $$[A^i(t,\vec{x}),\ \pi_j(t,\vec{y})] = i\, \delta^i_j\, \delta^{(3)}(x-y)$$

4.3 Massless Spin - 1 fields [Weinberg 8.2 - 8.5]

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

The Maxwell equation : $\partial_\mu F^{\mu\nu} = j^\nu$

It's well known $A^\mu(x)$ is not a directly observable

local gauge transformation $A'^\mu(x) = A^\mu(x) + \partial^\mu \Lambda(x)$.

The field strength tensor $F^{\mu\nu}$ is gauge inv. , but these doesn't hold for the interaction term

$$j_\mu A^\mu \rightarrow j_\mu (A^\mu + \partial^\mu \Lambda)$$

$$= j_\mu A^\mu + \partial^\mu [j_\mu(x) \Lambda(x)] - \Lambda \partial^\mu j_\mu$$

$\partial^\mu j_\mu = 0$ must be satisfied to leave the action invariant.

To calculate a quantity , one fixes the gauge.

Lorentz gauge	$\partial^\mu A_\mu = 0$
Coulomb gauge	$\nabla \cdot \vec{A} = 0$
...	

In the Coulomb gauge, $\nabla \cdot \vec{A} = 0$.

$$\partial_i F^{i0} = j^0$$

$$\Rightarrow \nabla \cdot \vec{E} = j^0 \qquad [\vec{E} = -\dot{\vec{A}} - \nabla A^0]$$

$$\Rightarrow \underbrace{\nabla \cdot \dot{\vec{A}}}_{=0 \text{, in Coulomb gauge}} + \nabla^2 A^0 = -j^0$$

Then $A^0(x)$ satisfies the Poisson equation $\nabla^2 A^0 = -j^0$

Therefore $A^0(x)$ is determined by the charge distribution $j^0(x)$.

$$A^0(t,\vec{x}) = \int d^3y \, \frac{j^0(t,\vec{y})}{4\pi|\vec{x}-\vec{y}|}$$

$\uparrow$

instantaneous Coulomb potential.

not a dynamical d.o.f. of the theory.

An arbitrary $A^\mu(x)$ can be made to satisfy the Coulomb gauge if

$$\nabla\cdot\vec{A}' = 0 = \nabla\cdot\vec{A} - \nabla^2\Lambda$$

$$\Rightarrow \nabla^2\Lambda = \nabla\cdot\vec{A} \quad , \quad [\text{ Poisson equation }]$$

Then
$$\vec{A}' = \vec{A} - \nabla\Lambda$$
$$= \vec{A} - \nabla\int d^3y \; G(x-y)\,\nabla\cdot A(t,y)$$

with
$$\nabla^2 G(x-y) = \delta^{(3)}(\vec{x}-\vec{y}) \quad , \quad G(x-y) = -\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{y}|}$$

One could arrive the same results by splitting $\vec{A}$ into transverse and longitudinal parts

$$\vec{A} = \vec{A}_\perp + \vec{A}_\parallel = (P_\perp + P_\parallel)\vec{A},$$

with $\nabla\cdot\vec{A}_\perp = 0$, $\nabla\times\vec{A}_\parallel = 0$

$$(P_\perp)_{ij} = \delta_{ij} - \partial_i\frac{1}{\nabla^2}\partial_j \quad ; \quad (P_\parallel)_{ij} = \partial_i\frac{1}{\nabla^2}\partial_j$$

$$A_\perp^i = (P_\perp)^{ij}A^j = A^i - \partial^i\frac{1}{\nabla^2}\partial^j A^j$$
$$= A^i - \partial^i\int d^3y \; G(x-y)\,\partial^j A^j$$

$\uparrow$

$$\frac{1}{\nabla^2}V(x) = \int d^3y \; G(x-y)\,V(y)$$

So the longitudinal d.o.f. is eliminated. $\vec{A}_\parallel = 0$

The coulomb gauge is very suitable for describing the photon fields with two transverse d.o.f.

Polarization vectors :

$$\left.\begin{aligned} \varepsilon^\mu(k,1) &= (0, \vec{\varepsilon}(k,1)) \\ \varepsilon^\mu(k,2) &= (0, \vec{\varepsilon}(k,2)) \end{aligned}\right\} \text{ transverse polarizations.}$$

$$\vec{k}\cdot\vec{\varepsilon}(k,1) = \vec{k}\cdot\vec{\varepsilon}(k,2) = 0,$$

$$\vec{\varepsilon}(k,\lambda)\cdot\vec{\varepsilon}(k,\lambda') = \delta_{\lambda\lambda'}$$

Quantization in the Coulomb gauge.

A^μ is purely transverse, $\vec{A} = \vec{A}_\perp$, $\vec{A}_\parallel = 0$

The electric field $\vec{E} = -\dot{\vec{A}} - \nabla A^0 = -\dot{\vec{A}}_\perp = \vec{E}_\perp$

Define the transverse delta function,

$$\delta^{(3)}_{\perp ij}(\vec{x}-\vec{y}) = (P_\perp)_{ij}\, \delta^{(3)}(\vec{x}-\vec{y})$$

Then the equal time commutation relations are

$$[\, A^i(t,\vec{x}),\ \pi^j(t,\vec{y})\,] = -i\,\delta^{(3)}_{\perp ij}(\vec{x}-\vec{y})$$

In the momentum space, the transverse delta function is

$$\delta^{(3)}_{\perp ij}(\vec{x}-\vec{y}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \left(\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2} \right)$$

The vector potential $A^\mu(t,\vec{x})$

$$A^i(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{\lambda=1}^{2} \varepsilon^i(k,\lambda) \left(a_{k,\lambda} e^{-ik\cdot x} + a^+_{k,\lambda} e^{ik\cdot x} \right)$$

$$\pi^i(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \sum_{\lambda=1}^{2} i\omega_k\, \varepsilon^i(k,\lambda) \left(a_{k,\lambda} e^{-ik\cdot x} - a^+_{k,\lambda} e^{ik\cdot x} \right)$$

$$[\, a_{p,\lambda},\ a^+_{q,\lambda'}\,] = (2\pi)^3 \delta_{\lambda\lambda'}\, \delta^{(3)}(\vec{p}-\vec{q}) \quad ; \quad [\, a_{p,\lambda}\ a_{q,\lambda'}\,] = [\, a^+_{p,\lambda},\ a^+_{q,\lambda'}\,] = 0$$

The equal time commutator

$$[\, A^i(t,x),\ \pi^j(t,y)\,] = \sum_{\lambda,\lambda'} \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} \int \frac{d^3q}{(2\pi)^3\sqrt{2\omega_q}}\ \varepsilon^i(p,\lambda)\, \varepsilon^j(q,\lambda')\, (i\omega_q)$$

$$\times \left\{ -[\, a_{p,\lambda},\ a^+_{q,\lambda'}\,] e^{-ip\cdot x + iq\cdot y} + [\, a^+_{p,\lambda},\ a_{q,\lambda'}\,] e^{ip\cdot x - iq\cdot y} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(-\frac{i}{2}\right) \underbrace{\sum_{\lambda=1}^{2} \varepsilon^i(p,\lambda)\, \varepsilon^j(p,\lambda)}_{=\ \delta_{ij} - \frac{p^i p^j}{|\vec{p}|^2}} \left[e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right]$$

$$= -i \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left(\delta_{ij} - \frac{p^i p^j}{|\vec{k}|^2} \right)$$

$$= -i\, \delta^{(3)}_{\perp ij}(\vec{x}-\vec{y})$$

The Hamiltonian and momentum operators are

$$H = \int d^3x \ N\left(|\vec{E}|^2 + |\vec{B}|^2 \right) = \int d^3k \ \omega_k \sum_{\lambda=1}^{2} a^{+}_{k,\lambda} a_{k,\lambda}$$

$$\vec{P} = \int d^3x \ N\left(\vec{E} \times \vec{B} \right) = \int d^3k \ \vec{k} \sum_{\lambda=1}^{2} a^{+}_{k,\lambda} a_{k,\lambda}$$

$\boxed{\text{Feynman propagator}}$ $D_F^{\mu\nu}(x-y) = \langle 0 | \, T A^\mu(x) A^\nu(y) \, | 0 \rangle$

$$D_F^{ij}(x-y) = \theta(x^0-y^0)\left(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2}\right) D(x-y) + \theta(y^0-x^0)\left(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2}\right) D(x-y)$$

$$= \left(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2}\right) D_F(x-y)$$

$$= \left(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2}\right) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\varepsilon} e^{-ik\cdot(x-y)}$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i(\delta^{ij} - \hat{k}^i\hat{k}^j)}{k^2+i\varepsilon} \qquad [\ \hat{k}^i = k^i/|\vec{k}|\ ]$$

Then $\quad D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i P^{\mu\nu}(\vec{k})}{k^2+i\varepsilon}$

with $\begin{cases} P^{ij}(\vec{k}) = \delta^{ij} - \hat{k}^i\hat{k}^j \\ P^{00} = P^{0i} = P^{0j} = 0 \end{cases}$

time-like polarization $\quad \varepsilon^\mu(\vec{k}, 0) = n^\mu = (1, 0, 0, 0)$

longitudinal polarization $\quad \varepsilon^\mu(\vec{k}, 3) = \left(0, \frac{\vec{k}}{|\vec{k}|}\right) = \frac{k^\mu - (n\cdot k) n^\mu}{[(k\cdot n)^2 - k^2]^{1/2}}$

$$\varepsilon(\vec{k},3)\cdot\varepsilon(\vec{k},3) = -1, \quad n\cdot\varepsilon(\vec{k},3) = 0$$

The completeness relation:

$$\sum_{\lambda=0}^{3} \varepsilon_\mu(k,\lambda)\, \varepsilon^*_\nu(k,\lambda) = g_{\mu\nu}$$

then

$$P_{\mu\nu}(\vec{k}) = \sum_{\lambda=1}^{2} \varepsilon_\mu(k,\lambda)\, \varepsilon^*_\nu(k,\lambda)$$

$$= -g_{\mu\nu} - \frac{k_\mu k_\nu - (k_\mu n_\nu + k_\nu n_\mu) k\cdot n + n_\mu n_\nu k^2}{(k\cdot n)^2 - k^2}$$

$$= -g_{\mu\nu} - \frac{n_\mu n_\nu k^2}{(k\cdot n)^2 - k^2} - \frac{k_\mu k_\nu - (k_\mu n_\nu + k_\nu n_\mu) k\cdot n}{(k\cdot n)^2 - k^2}$$

so the Feynman propagator:

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\varepsilon} P^{\mu\nu}(k)\, e^{-ik\cdot(x-y)}$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \left[\frac{-ig_{\mu\nu}}{k^2+i\varepsilon} - \frac{ik^2 n_\mu n_\nu}{(k^2+i\varepsilon)[(k\cdot n)^2-k^2]} + \text{extra} \right]$$

$$\Downarrow$$

$$= -i\frac{n_\mu n_\nu}{(k^0)^2-k^2} = -i\frac{n_\mu n_\nu}{|\vec{k}|^2}$$

$$\equiv D_F^{\mu\nu,\,\mathrm{eff}}(x-y) + D_F^{\mu\nu,\,\mathrm{coul}}(x-y) + D_F^{\mu\nu,\,\mathrm{extra}}(x-y)$$

with
$$D_F^{\mu\nu,\,\mathrm{Coul}}(x-y) = -i\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{n_\mu n_\nu}{|\vec{k}|^2}$$

$$= -i\,\delta_{\mu 0}\delta_{\nu 0} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{1}{|\vec{k}|^2} \int \frac{dk^0}{(2\pi)} e^{-ik^0\cdot(x^0-y^0)}$$

$$= -i\,\delta_{\mu 0}\delta_{\nu 0}\,\delta(x^0-y^0)\frac{1}{4\pi|\vec{x}-\vec{y}|}$$

Covariant photon propagator.

$$D_F^{\mu\nu,\,\mathrm{eff}}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{-ig_{\mu\nu}}{k^2+i\varepsilon}$$

Consider external source

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

$$A^0(t, \vec{x}) = \frac{1}{4\pi} \int d^3y \, \frac{j^0(t, y)}{|\vec{x} - \vec{y}|}$$

The electric field strength has an additional longitudinal part:

$$\vec{E} = \underbrace{-\dot{\vec{A}}}_{\equiv \vec{E}_\perp} \underbrace{- \nabla A^0}_{\equiv \vec{E}_\parallel} \qquad \nabla \cdot \vec{E}_\perp = 0 \; ; \quad \nabla \times \vec{E}_\parallel = 0 .$$

Then $\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$$= \frac{1}{2} (|\vec{E}|^2 - |\vec{B}|^2)$$

$$= \frac{1}{2} (\vec{E}_\parallel^2 + \vec{E}_\perp^2 - \vec{B}^2) + \vec{E}_\parallel \cdot \vec{E}_\perp$$

where the mixed term

$$\vec{E}_\parallel \cdot \vec{E}_\perp = -(\nabla \cdot A^0) \cdot \vec{E}_\perp$$

$$= -\nabla \cdot (A^0 \vec{E}_\perp) + A^0 \nabla \cdot \vec{E}_\perp$$

$$= 0$$

The purely transverse canonically conjugate field

$$\vec{\pi}_\perp = -\frac{\partial \mathcal{L}}{\partial \dot{\vec{A}}} = \frac{\partial \mathcal{L}}{\partial \vec{E}_\perp} = \vec{E}_\perp$$

leading to

$$\mathcal{H} = -\vec{\pi}_\perp \cdot \dot{\vec{A}} - \mathcal{L}$$

$$= \vec{E}_\perp^2 - \frac{1}{2} (\vec{E}_\parallel^2 + \vec{E}_\perp^2 - \vec{B}^2) + j^0 A^0 - \vec{j} \cdot \vec{A}$$

$$= \underbrace{\frac{1}{2} \vec{E}_\perp^2 + \frac{1}{2} (\nabla \times \vec{A})^2}_{\mathcal{H}_0} \underbrace{- \frac{1}{2} \vec{E}_\parallel^2 + j^0 A^0 - \vec{j} \cdot \vec{A}}_{= V}$$

with $-\frac{1}{2} \vec{E}_\parallel^2 = -\frac{1}{2} (\nabla A^0) \cdot (\nabla A^0)$

$$= -\frac{1}{2} \underset{\overset{\shortparallel}{0}}{\nabla \cdot (A^0 \nabla A^0)} + \frac{1}{2} A^0 \underbrace{\nabla^2 A^0}_{= -j^0} \quad \text{Poisson equation}$$

$$= -\frac{1}{2} A^0 j^0$$

Then $V = \int d^3x \, [\, \frac{1}{2} A^0 j^0 - \vec{j} \cdot \vec{A} \,]$

$\equiv V_{Coul} + V^{\perp}$

$V_{Coul} = \frac{1}{2} \int d^3x \, d^3y \; j_0(t, \vec{x}) \, \frac{1}{4\pi |\vec{x} - \vec{y}|} \, j_0(t, \vec{y})$ Coulomb interaction

In Heinsenberg picture $A^0 \neq 0$

In Interaction picture $A^0_I = 0$

Then $V_I^{\perp} = \int d^3x \; j^{\mu} A_{\mu}$

$$\langle 0 | \, T\exp[-i \int dt \, V_I(t)] \, | 0 \rangle$$

$$= 1 - i \int dt \; V_{Coul} + \frac{(-i)^2}{2!} \int d^4x \, d^4y \; T\,(j_{\mu}(x) A^{\mu}(x) \, j_{\nu}(y) A^{\nu}(y))$$

$$= 1 \underbrace{- \frac{i}{2} \int d^4x \, d^4y \; \delta(x^0 - y^0) \, \frac{j^0(x) \, j^0(y)}{4\pi |\vec{x} - \vec{y}|}}_{\text{Cancelled with } D_F^{\mu\nu,\,Coul}(x-y)} - \frac{1}{2} \int d^4x \, d^4y \; D_F^{\mu\nu}(x-y) \;\; N[\, j_{\mu}(x) j_{\nu}(y) \,]$$

The terms in $D_F^{\mu\nu,\,extra}(x-y)$ are propotional to k^{μ} or k^{ν}. The EM field coulpes to conserved currents

$$\partial^{\mu} j_{\mu}(x) = 0 \quad \text{or} \quad k^{\mu} j_{\mu}(x) = 0 .$$

So the extra term does not contribute in the calculation and it can be simply left out.