6. Quantum electrodynamics (QED)

The Lagrangian for QED.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}(\not{\partial} + ie\not{A})\psi - m\bar{\psi}\psi$$

A photon propagator is represented with a squiggly line:

$$= \frac{-i}{p^2 + i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]. \tag{13.10}$$

Unless we are explicitly checking gauge invariance, we will usually work in Feynman gauge, $\xi = 1$, where the propagator is

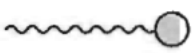
$$= \frac{-ig_{\mu\nu}}{p^2 + i\varepsilon} \quad \text{(Feynman gauge)}. \tag{13.11}$$

A spinor propagator is a solid line with an arrow:

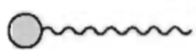
$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}. \tag{13.12}$$

The arrow points to the right for particles and to the left for antiparticles. For internal lines, the arrow points with momentum flow.

External photon lines get polarization vectors:



$$= \epsilon_\mu(p) \quad \text{(incoming)}, \tag{13.13}$$



$$= \epsilon_\mu^\star(p) \quad \text{(outgoing)}. \tag{13.14}$$

Here the blob means the rest of the diagram.

External fermion lines get spinors, with u spinors for electrons and v spinors for positrons.

$$= u^s(p), \tag{13.15}$$

$$= \bar{u}^s(p), \tag{13.16}$$

$$= \bar{v}^s(p), \tag{13.17}$$

$$= v^s(p). \tag{13.18}$$

External spinors are on-shell (they are forced to be on-shell by LSZ). So, for external spinors we can use the equations of motion:

$$(\not{p} - m)u^s(p) = \bar{u}^s(p)(\not{p} - m) = 0, \tag{13.19}$$

$$(\not{p} + m)v^s(p) = \bar{v}^s(p)(\not{p} + m) = 0, \tag{13.20}$$

which will simplify a number of calculations.

Expanding the Lagrangian,

$\boxed{\text{The gauge principle}}$ [Peskin 15.1]

QED : $\mathcal{L}_{QED} = \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} + \mathcal{L}_{int}$

$$= \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^\mu\psi A_\mu$$

$\mathcal{L}_{Dirac}$ is invariant under $\psi(x) \to e^{i\alpha}\psi(x)$ [internal symmetry]

[α is not position-dependent]

The transformation is no longer a symmetry for general $\alpha = \alpha(x)$

↑

different phase rotations at different spacetime point

See: $m\bar{\psi}(x)\psi(x)$ is invariant under $\psi(x) \to e^{i\alpha(x)}\psi(x)$

$$m\bar{\psi}(x)\psi(x) \to m\bar{\psi}(x)e^{-i\alpha(x)}e^{i\alpha(x)}\psi(x)$$

$$= m\bar{\psi}(x)\psi(x)$$

The kinetic term :

$$\partial_\mu\psi(x) \to \partial_\mu[e^{i\alpha(x)}\psi(x)] = e^{i\alpha(x)}[i\partial_\mu\alpha(x)]\psi(x) + \underbrace{e^{i\alpha(x)}\partial_\mu\psi(x)}_{\text{the 2nd term is the problem}}$$

We want to construct a theory which is invariant for a general $\alpha(x)$, i.e. local $U(1)$ symmetry.

local symmetries → Interactions

local $U(1)$ symmetries → Electromagnetic interaction

local $U(n)$ symmetries → non-Abelian gauge interaction

We will "rederive" QED from the requirement of local $U(1)$ symmetry.

$$\mathcal{L} \text{ is invariant under } \psi(x) \to e^{i\alpha(x)}\psi(x)$$

"gauge transformation"

Consider the derivative along some direction n^μ

$$n^\mu \partial_\mu \psi(x) = \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon n) - \psi(x)}{\varepsilon}$$

If we can rotate $\psi(x+\varepsilon n)$ and $\psi(x)$ independently, $n\cdot\partial\psi(x)$ does not have a definite meaning.

Introducing a new derivative D_μ

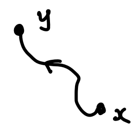
$$D_\mu \psi(x) \longrightarrow e^{i\alpha(x)} D_\mu \psi(x)$$

To do this, assume we have an object $U(y,x)$ that transforms as

$$U(y,x) \longrightarrow e^{i\alpha(y)}\, U(y,x)\, e^{-i\alpha(x)}$$

$\Uparrow$

"transport" the gauge transformation from $x \to y$



$$U(y,x)\,\psi(x) \to e^{i\alpha(y)}\, U(y,x)\, e^{-i\alpha(x)}\, e^{i\alpha(x)}\, \psi(x)$$

$$= e^{i\alpha(y)}\, U(y,x)\, \psi(x)$$

"transforming as $\psi(y)$"

$\psi(y)$ and $U(y,x)\,\psi(x)$ have the same transformation properties,

$\psi(y) - U(y,x)\,\psi(x)$ is well-defined.

Now take $y = x + \varepsilon n$, and define

$$n^\mu D_\mu \psi(x) = \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\,\psi(x)}{\varepsilon}$$

By construction.

$$n^\mu D_\mu \psi(x) \longrightarrow \lim_{\varepsilon \to 0} \frac{e^{i\alpha(x+\varepsilon n)}\psi(x+\varepsilon n) - e^{i\alpha(x+\varepsilon n)}\, U(x+\varepsilon n, x)\,\psi(x)}{\varepsilon}$$

$$= e^{i\alpha(x)}\, n\cdot D\psi(x)$$

Then $\mathcal{L}_{Dirac}$ is invariant under the gauge transformation.

We now want to construct $U(y,x)$ explicitly.

$$U(y,x) = e^{i\phi(x,y)} \quad \text{a pure phase.}$$

Begin infinitesimally,

$$U(x+\varepsilon n, x) = 1 - i\varepsilon n^{\mu} e A_{\mu}(x) + O(\varepsilon^2)$$

e: a constant

A_{μ}: a real vector field

Under the gauge transformation

$$U(x+\varepsilon n, x) \rightarrow e^{i\alpha(x+\varepsilon n)} U(x+\varepsilon n, x) e^{i\alpha(x)}$$

$$\Rightarrow \quad 1 - ie\,\varepsilon n^{\mu} A_{\mu}(x) \rightarrow e^{i\alpha + i\varepsilon n^{\mu}\partial_{\mu}\alpha(x)} (1 - ie\,\varepsilon n^{\mu} A_{\mu}(x))\, e^{-i\alpha(x)}$$

$$= e^{i\alpha(x)} (1 - ie\,\varepsilon n^{\mu} A_{\mu})\, e^{-i\alpha(x)}$$

$$+ e^{i\alpha(x)} (i\varepsilon n^{\mu}\partial_{\mu}\alpha)\, e^{-i\alpha(x)}$$

$$= 1 - ie\,\varepsilon n^{\mu} A_{\mu}(x) + i\varepsilon n^{\mu}\partial_{\mu}\alpha + O(\varepsilon^2)$$

$$\Rightarrow \quad A^{\mu}(x) \longrightarrow A^{\mu}(x) - \frac{1}{e}\partial^{\mu}\alpha$$

For the covariant derivative D_{μ}, we have

$$n^{\mu} D_{\mu}\psi(x) = \lim_{\varepsilon\to 0} \frac{\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\psi(x)}{\varepsilon}$$

$$= \lim_{\varepsilon\to 0} \frac{\psi(x) + \varepsilon n^{\mu}\partial_{\mu}\psi - [1 - ie\,\varepsilon n^{\mu} A_{\mu}(x)]\psi(x)}{\varepsilon}$$

$$= n^{\mu}\partial_{\mu}\psi(x) + ie\, n^{\mu} A_{\mu}\psi(x)$$

$$\Rightarrow \quad D^{\mu} = \partial_{\mu} + ie A_{\mu}$$

We want $A_{\mu}(x)$ to be a dynamical field, and then we require a kinetic term:

a gauge invariant term that depends on A_{μ} and its derivatives, but not on ψ.

We note that $D_\mu(D_\nu \psi)$ transforms as ψ

$$
\begin{aligned}
D_\mu(D_\nu \psi) &= (\partial_\mu + ie A_\mu)(\partial_\nu + ie A_\nu)\psi \\
&\to (\partial_\mu + ie A_\mu - i\partial_\mu \alpha)(\partial_\nu + ie A_\nu - i\partial_\nu \alpha) e^{i\alpha(x)} \psi(x) \\
&= (\partial_\mu + ie A_\mu - i\partial_\mu \alpha) e^{i\alpha(x)} D_\nu \psi(x) \\
&= e^{i\alpha(x)} D_\mu(D_\nu \psi)
\end{aligned}
$$

So we define

$$
\begin{aligned}
[D_\mu, D_\nu] &= [\partial_\mu + ie A_\mu, \partial_\nu + ie A_\nu] \\
&= (\partial_\mu + ie A_\mu)(\partial_\nu + ie A_\nu) - (\partial_\nu + ie A_\nu)(\partial_\mu + ie A_\mu) \\
&= ie(\partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \partial_\nu - A_\nu \partial_\mu) \\
&= ie[(\partial_\mu A_\nu) + A_\nu \partial_\mu - (\partial_\nu A_\mu) - A_\mu \partial_\nu + A_\mu \partial_\nu - A_\nu \partial_\mu] \\
&= ie[(\partial_\mu A_\nu) - (\partial_\nu A_\mu)] \\
&\equiv ie F_{\mu\nu}
\end{aligned}
$$

We have that $F_{\mu\nu}\psi$ transforms as ψ, so that $F_{\mu\nu}$ is gauge invariant.

Some comments:

1° A $U(1)$ local symmetry leads to the field $A_\mu(x)$.

2° No mass term is allowed for A_μ; gauge symmetry is broken

3° It's possible to construct theories using other gauge invariant terms

$$\varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}, \quad (F_{\mu\nu} F^{\mu\nu})^2, \quad \bar{\psi} F_{\mu\nu} \gamma^\mu \gamma^\nu \psi$$

4° Some of the terminology and constructs associated with gauge theories

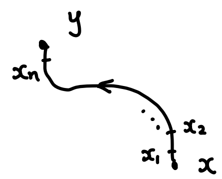
$$
\left.
\begin{array}{lcl}
U(y,x) & : & \text{Parallel transport} \\
A_\mu(x) & : & \text{Connection} \\
F_{\mu\nu}(x) & : & \text{Curvature} \\
D_\mu(x) & : & \text{Covariant derivative}
\end{array}
\right\} \quad \text{fibre bundle}
$$

5° Gauge symmetry is not really a symmetry.

The phase of ψ does not carry physical information

A redundancy in our variables

6° The explicit form of $U(y,x)$: choose a path γ from x to y



[Peskin 15.3]

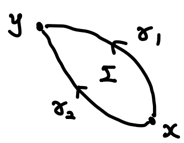
Split the path γ into infinitesimal segments:

$$U_\gamma(y,x) = U(y,x_n)\,U(x_n,x_{n-1})\cdots U(x_1,x)$$

Since $\quad U(x_1,x) \approx \exp[-ie\,A_\mu(x)\,(x_1-x)^\mu]$

$$\Rightarrow \quad U_\gamma(y,x) = \exp\left[-ie\int_\gamma A_\mu(x)\,dx^\mu\right]$$

Note: $U_\gamma(y,x)$ is not necessarily path-independent : $U_{\gamma_1}(y,x) \neq U_{\gamma_2}(y,x)$



Σ: the enclosed area

γ_1, γ_2 : paths from x to y

Let $\quad U_P(x,x) = U_{-\gamma_2}(y,x)\,U_{\gamma_1}(y,x) \qquad$ [Wilson loop]

$$= \exp\left[-ie\oint_P A_\mu(x)\,dx^\mu\right]$$

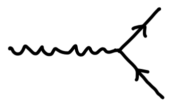
$$= \exp\left[-i\frac{e}{2}\int_\Sigma F_{\mu\nu}(x)\,d\sigma^{\mu\nu}\right] \qquad \text{Stoke's theorem}$$

Σ is a surface that spans the loop P

$d\sigma^{\mu\nu}$ is an area element on this surface

The Wilson loop is associated with phenomena such as the Aharonov-Bohm effect, and Berry's phase.

The interaction $\mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu\psi A_\mu$

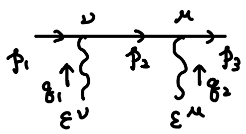


$= -ie\gamma^\mu$

The $\gamma^\mu = \gamma^\mu_{\alpha\beta}$ will always sandwiched between spinors

$$\bar{u}\gamma^\mu u = \bar{u}_\alpha \gamma^\mu_{\alpha\beta} u_\beta$$

If there is an internal fermion line, then

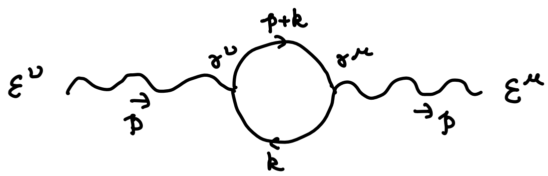


$$= (-ie)^2\, \bar{u}(p_3)\gamma^\mu \frac{i(\not{p}_2+m)}{p_2^2-m^2+i\varepsilon}\gamma^\nu u(p_1)\, \varepsilon_\mu(q_2)\, \varepsilon_\nu(q_1) \qquad [\, q_1^\mu = p_2^\mu - p_1^\mu;\ q_2^\mu = p_3^\mu - p_2^\mu\,]$$

add in the spinor indices:

$$\bar{u}(p_3)\gamma^\mu(\not{p}_2+m)\gamma^\nu u(p_1) = \bar{u}_\alpha(p_3)\gamma^\mu_{\alpha\beta}(\not{p}_2+m)_{\beta\gamma}\gamma^\nu_{\gamma\delta}u_\delta(p_1)$$

If we tie the ends of the diagram above together, we have a loop.



In the loop, any possible intermediate states are allowed, we must integrate ove the momenta as well as sum over their possible spins.

$u_\delta \bar{u}_\alpha$ gets replaced by a propagator : $(\not{p}_2 + m)_{\delta\alpha} = \sum_s u^s_\delta \bar{u}^s_\alpha$

So the loop evaluates to

$$i\mathcal{M} = -(-ie)^2\int\frac{d^4k}{(2\pi)^4}\,\varepsilon^*_\mu(p)\,\varepsilon_\nu(p)\left[\gamma^\mu_{\alpha\beta}\frac{i(\not{p}+\not{k}+m)_{\beta\gamma}}{(p+k)^2-m^2+i\varepsilon}\gamma^\nu_{\gamma\delta}\frac{i(\not{k}+m)_{\delta\alpha}}{k^2-m^2+i\varepsilon}\right]$$

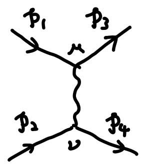
Contracting all spinor indices,

$$i\mathcal{M} = e^2\,\varepsilon^*_\mu(p)\,\varepsilon_\nu(p)\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^\mu\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}\gamma^\nu\frac{i(\not{k}+m)}{k^2-m^2}\right]$$

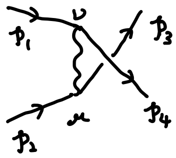
a trace of spinor indices.

The spinor matrices are always multiplied together in the direction opposite to the particle flow.

Consider Moller scattering ($e^-e^- \to e^-e^-$) at tree level.

$i\mathcal{M}_t =$ 

$= \pm(-ie)^2\, \bar{u}(p_3)\gamma^\mu u(p_1) \dfrac{-ig_{\mu\nu}}{(p_1-p_3)^2}\, \bar{u}(p_4)\gamma^\nu u(p_2)$

$i\mathcal{M}_u =$ 

$= \pm(-ie)^2\, \bar{u}(p_3)\gamma^\mu u(p_2) \dfrac{-ig_{\mu\nu}}{(p_1-p_4)^2}\, \bar{u}(p_4)\gamma^\nu u(p_1)$

What sign should each diagram have ?

The corresponding Green function.

$$G(x_1\ x_2\ x_3\ x_4) = \langle\Omega|\, T\{\psi(x_1)\bar{\psi}(x_3)\psi(x_2)\bar{\psi}(x_4)\}\,|\Omega\rangle$$

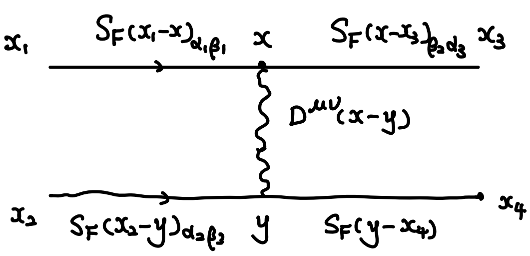
At order e^2

$$(-ie)^2\int d^4x\, d^4y\ \langle 0|\, T\{\psi_{\alpha_1}(x_1)\bar{\psi}_{\alpha_3}(x_3)\psi_{\alpha_2}(x_2)\bar{\psi}_{\alpha_4}(x_4)\ \bar{\psi}_{\beta_1}(x)\gamma^\mu_{\beta_1\beta_2}A_\mu(x)\psi_{\beta_2}(x)$$
$$\times\ \bar{\psi}_{\beta_3}(y)\gamma^\nu_{\beta_3\beta_4}A_\nu(y)\psi_{\beta_4}(y)\ \}\,|0\rangle$$

For $i\mathcal{M}_t$

$$(-ie)^2\,\gamma^\mu_{\beta_1\beta_2}\gamma^\nu_{\beta_3\beta_4}\int d^4x\, d^4y$$
$$\times\ \langle 0|\, T\{\psi_{\alpha_1}(x_1)\bar{\psi}_{\alpha_3}(x_3)\psi_{\alpha_2}(x_2)\bar{\psi}_{\alpha_4}(x_4)\bar{\psi}_{\beta_1}(x)\psi_{\beta_2}(x)\bar{\psi}_{\beta_3}(y)\psi_{\beta_4}(y)\ \underbrace{A_\mu(x)A_\nu(y)}\}\,|0\rangle$$

$$\Rightarrow\ \underbrace{A_\mu(x)A_\nu(y)}\ \underbrace{\psi_{\alpha_1}(x_1)\bar{\psi}_{\beta_1}(x)}\ \underbrace{\psi_{\beta_2}(x)\bar{\psi}_{\alpha_3}(x_3)}\ \underbrace{\psi_{\alpha_2}(x_2)\bar{\psi}_{\beta_3}(y)}\ \underbrace{\psi_{\beta_4}(y)\bar{\psi}_{\alpha_4}(x_4)}$$

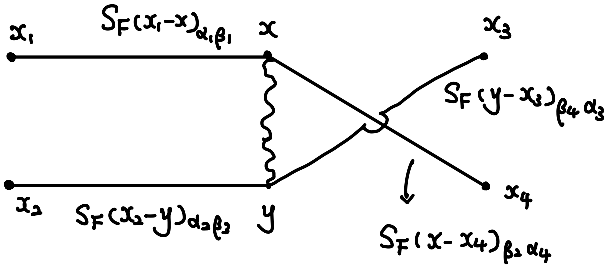


For $i\mathcal{M}_u$

$$(-ie)^2 \gamma^{\mu}_{\beta_1\beta_2} \gamma^{\nu}_{\beta_3\beta_4} \int d^4x\, d^4y$$

$$\times \langle 0| T\{ \psi_{\alpha_1}(x_1) \bar{\psi}_{\alpha_3}(x_3) \psi_{\alpha_2}(x_2) \bar{\psi}_{\alpha_4}(x_4) \bar{\psi}_{\beta_1}(x) \psi_{\beta_2}(x) \bar{\psi}_{\beta_3}(y) \psi_{\beta_4}(y) A_{\mu}(x) A_{\nu}(y) \} |0\rangle$$

$$\Rightarrow - \underbrace{A_{\mu} A_{\nu}}\ \underbrace{\psi_{\alpha_1}(x_1) \bar{\psi}_{\beta_1}(x)}\ \underbrace{\psi_{\beta_4}(y) \bar{\psi}_{\alpha_3}(x_3)}\ \underbrace{\psi_{\alpha_2}(x_2) \bar{\psi}_{\beta_3}(y)}\ \underbrace{\psi_{\beta_2}(x) \bar{\psi}_{\alpha_4}(x_4)}$$



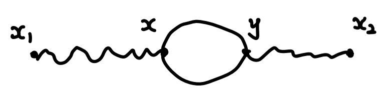
$\Rightarrow$ u-channel has a minus sign out front.

$$\mathcal{M} = \mathcal{M}_t + \mathcal{M}_u = e^2 \left\{ \frac{[\bar{u}(p_3)\gamma^{\mu}u(p_1)][\bar{u}(p_4)\gamma_{\mu}u(p_2)]}{(p_1-p_3)^2} - \frac{[\bar{u}(p_4)\gamma^{\mu}u(p_1)][\bar{u}(p_3)\gamma_{\mu}u(p_2)]}{(p_1-p_4)^2} \right\}$$

The over-all sign is unphysical phase, but the relative sign of the t-channel and u-channel is important.

$$|\mathcal{M}|^2 = |\mathcal{M}_t + \mathcal{M}_u|^2$$

For the fermion loop.



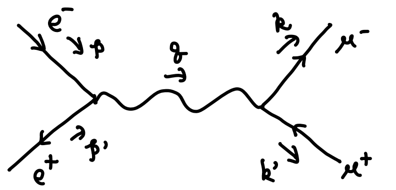
$$G(x_1, x_2) = (-ie)^2 \int d^4x\, d^4y \ \langle 0| T\{ A_{\mu}(x_1) A_{\nu}(x_2) \bar{\psi}(x) \cancel{A}(x) \psi(x) \bar{\psi}(y) \cancel{A}(y) \psi(y) \} |0\rangle$$

$$\Rightarrow \underbrace{A_{\mu}(x_1) A_{\alpha}(x)}\ \underbrace{A_{\nu}(x_2) A_{\beta}(y)}\ \underbrace{\bar{\psi}(x) \psi(x) \bar{\psi}(y) \psi(y)}$$

$$= D_{\mu\alpha}(x_1-x)\, D_{\nu\beta}(x_2-y)\ (-1)\ \mathrm{Tr}[\psi_x \bar{\psi}_y\ \psi(y) \bar{\psi}(x)]$$

$$= D_{\mu\alpha} D_{\nu\beta}\ (-1)\ \mathrm{Tr}[S_F(x-y)\, S_F(y-x)]$$

6.1 $\quad e^+e^- \to \mu^+\mu^-$:



$$= \bar v^{s'}(p')(-ie\gamma^\mu)u^s(p)\,\frac{-ig_{\mu\nu}}{q^2}\,\bar u^r(k)(-ie\gamma^\nu)v^{r'}(k')$$

To compute the cross section, one needs $|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^*$

$$(\bar v\gamma^\mu u)^* = u^+(\gamma^\mu)^+(\gamma^0)^+ v = u^+(\gamma^\mu)^+\gamma^0 v = u^+(\gamma^0)^2(\gamma^\mu)^+\gamma^0 v = \bar u\gamma^\mu v$$

$$\Rightarrow \quad \mathcal{M} = \frac{e^2}{q^2}\,\bar v\gamma^\mu u\;\bar u\gamma_\mu v \quad , \quad \mathcal{M}^* = \frac{e^2}{q^2}\,\bar u\gamma^\nu v\;\bar v\gamma_\nu u$$

$$\Rightarrow \quad |\mathcal{M}|^2 = \frac{e^4}{q^4}\left[\bar v(p')\gamma^\mu u(p)\,\bar u(p)\gamma^\nu v(p')\right]\left[\bar u(k)\gamma_\mu v(k')\,\bar v(k')\gamma_\nu u(k)\right]$$

Let's consider e^+, e^-, μ^+, μ^- are unpolarized, the cross section is an average over e^+, e^- spin s, s' and a sum over the muon spin r and r'.

Then we want to compute

$$\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'} \sum_r \sum_{r'} |\mathcal{M}(s,s' \to r,r')|^2$$

The spin sum

$$\sum_s u^s(p)\,\bar u^s(p) = \not p + m \;; \qquad \sum_s v^s(p)\,\bar v^s(p) = \not p - m .$$

So

$$\sum_{s,s'} \bar v^{s'}_a(p')\,\gamma^\mu_{ab}\,u^s_b(p)\,\bar u^s_c(p)\,\gamma^\nu_{cd}\,v^{s'}_d(p') = (\not p - m)_{da}\,\gamma^\mu_{ab}\,(\not p + m)_{bc}\,\gamma^\nu_{cd}$$

$$= \mathrm{Tr}\left[(\not p' - m)\gamma^\mu(\not p + m)\gamma^\nu\right]$$

Then we have

$$\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\,\mathrm{Tr}\left[(\not p' - m_e)\gamma^\mu(\not p + m_e)\gamma^\nu\right]\,\mathrm{Tr}\left[(\not k + m_\mu)\gamma_\mu(\not k' - m_\mu)\gamma_\nu\right]$$

$\boxed{\text{Trace Technology}}$

$$\mathrm{Tr}\,\gamma^\mu = \mathrm{Tr}\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} = 0 .$$

For an abitrary odd number of γ matrices:

$$\mathrm{Tr}\,\gamma^\mu \underset{\substack{\uparrow \\ (\gamma^5)^2 = 1}}{=} \mathrm{Tr}\,\gamma^5\gamma^5\gamma^\mu \underset{\substack{\uparrow \\ \{\gamma^\mu,\gamma^5\} = 0}}{=} -\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^5 = -\mathrm{Tr}\,\gamma^5\gamma^5\gamma^\mu = -\mathrm{Tr}\,\gamma^\mu \quad \Rightarrow \mathrm{Tr}\,\gamma^\mu = 0$$

For n γ-matrices, one gets $(-1)^n$ as we move γ^5 all the way to right.

so the trace vanishes if n is odd.

For two γ matrices.

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(2g^{\mu\nu}\mathbb{1}_{4\times4} - \gamma^\nu\gamma^\mu)$$

$$= 8g^{\mu\nu} - \mathrm{Tr}(\gamma^\nu\gamma^\mu)$$

$$= 8g^{\mu\nu} - \mathrm{Tr}(\gamma^\mu\gamma^\nu)$$

$$\Rightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

For four γ matrices:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = \mathrm{Tr}(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma)$$

$$= \mathrm{Tr}(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu 2g^{\mu\rho}\gamma^\sigma + \gamma^\nu\gamma^\rho 2g^{\mu\sigma} - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu)$$

$$\Rightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\nu\rho}g^{\mu\sigma})$$

For traces involving γ^5

$$\mathrm{Tr}\,\gamma^5 = \mathrm{Tr}(\gamma^0\gamma^0\gamma^5) = -\mathrm{Tr}(\gamma^0\gamma^5\gamma^0) = -\mathrm{Tr}(\gamma^0\gamma^0\gamma^5) = -\mathrm{Tr}\,\gamma^5$$

The same trick works for $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^5)$, if one inserts γ^α $(\alpha \neq \mu,\nu)$

$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = 0$ unless $(\mu\nu\rho\sigma)$ is some permutation of (0123)

Summary:

$$\begin{aligned}
\mathrm{tr}(\mathbf{1}) &= 4 \\
\mathrm{tr}(\text{any odd \# of } \gamma\text{'s}) &= 0 \\
\mathrm{tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} \\
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\
\mathrm{tr}(\gamma^5) &= 0 \\
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^5) &= 0 \\
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) &= -4i\epsilon^{\mu\nu\rho\sigma}
\end{aligned}$$

When two γ matrices inside a trace are dotted together

$$\gamma^\mu \gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2} g_{\mu\nu}\{\gamma^\mu \gamma^\nu\} = g_{\mu\nu}g^{\mu\nu} = 4.$$

Similarly,

$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$$

$$\cdots$$

Now return to $|\mathcal{M}|^2$.

$$\mathrm{Tr}[(\not{p}' - m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu] = \mathrm{Tr}[\not{p}'\gamma^\mu\not{p}\gamma^\nu] - m_e^2\,\mathrm{Tr}[\gamma^\mu\gamma^\nu]$$

$$= 4p'^\mu p^\nu + 4p'^\nu p^\mu - 4g^{\mu\nu}(p\cdot p' + m_e^2)$$

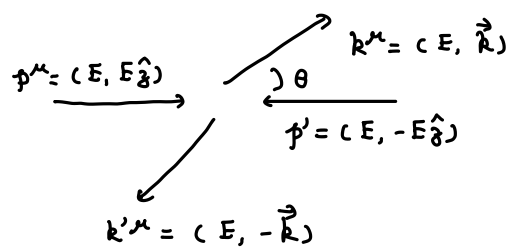
$$\mathrm{Tr}[(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu] = 4k_\mu k'_\nu + 4k_\nu k'_\mu - 4g_{\mu\nu}(k\cdot k' + m_\mu^2)$$

Let's set $m_e = 0$, $m_e \sim 500\,\mathrm{eV}$, $m_\mu \sim 100\,\mathrm{MeV}$ $m_e/m_\mu \sim 1/200$.

Then

$$\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{8e^4}{q^4}[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')]$$

In the center of mass frame of e^+, e^-



$$|\vec{k}| = \sqrt{E^2 - m_\mu^2}$$

$$\vec{k}\cdot\hat{z} = |k|\cos\theta.$$

Then

$$q^2 = (p+p')^2 = 4E^2 \qquad p\cdot p' = 2E^2$$

$$p\cdot k = p'\cdot k' = E^2 - E|\vec{k}|\cos\theta \qquad p\cdot k' = p'\cdot k = E^2 + E|\vec{k}|\cos\theta$$

$$\Rightarrow \frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = e^4\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

Then the cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{cm}^2}\frac{|\vec{k}|}{16\pi^2 E_{cm}}\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 \qquad [\, E_{cm} = 2E \,]$$

Integrating over $d\Omega$

$$\sigma_{total} = \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right)$$

In the high-energy limit $E \gg m_\mu$.

$$\sigma_{total} \xrightarrow{E \gg m_\mu} \frac{4\pi\alpha^2}{3E_{cm}^2}$$

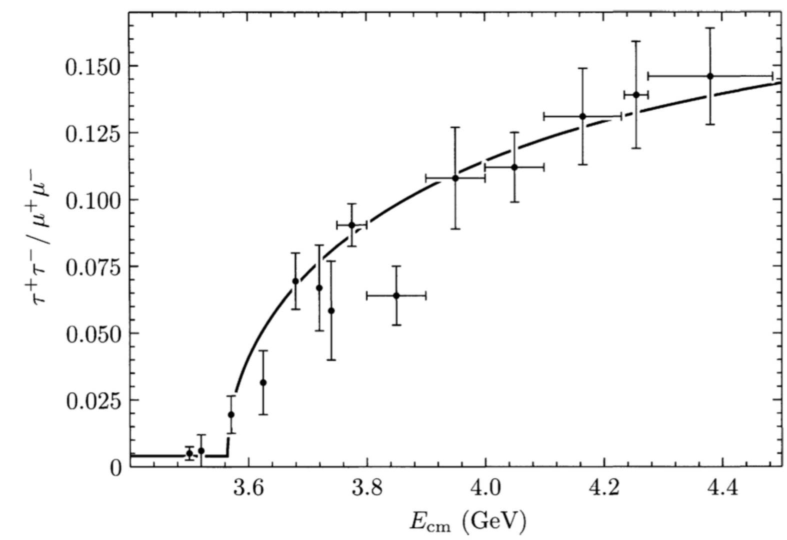
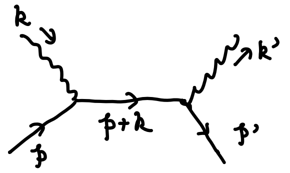
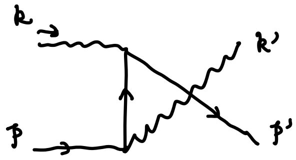


Figure 5.2. The ratio $\sigma(e^+e^- \to \tau^+\tau^-)/\sigma(e^+e^- \to \mu^+\mu^-)$ of measured cross sections near the threshold for $\tau^+\tau^-$ pair-production, as measured by the DELCO collaboration, W. Bacino, et. al., *Phys. Rev. Lett.* **41,** 13 (1978). Only a fraction of τ decays are included, hence the small overall scale. The curve shows a fit to the theoretical formula (5.13), with a small energy-independent background added. The fit yields $m_\tau = 1782^{+2}_{-7}$ MeV.

1. Draw the diagram(s) for the desired process.
2. Use the Feynman rules to write down the amplitude $\mathcal{M}$.
3. Square the amplitude and average or sum over spins, using the completeness relations (5.3). (For processes involving photons in the final state there is an analogous completeness relation, derived in Section 5.5.)
4. Evaluate traces using the trace theorems (5.5); collect terms and simplify the answer as much as possible.
5. Specialize to a particular frame of reference, and draw a picture of the kinematic variables in that frame. Express all 4-momentum vectors in terms of a suitably chosen set of variables such as E and θ.
6. Plug the resulting expression for $|\mathcal{M}|^2$ into the cross-section formula (4.79), and integrate over phase-space variables that are not measured to obtain a differential cross section in the desired form. (In our case these integrations were over the constrained momenta $\mathbf{k}'$ and $|\mathbf{k}|$, and were performed in the derivation of Eq. (4.84).)

6.2 Compton Scattering $e^- \gamma \to e^- \gamma$ [Peskin 5.5]





$$i\mathcal{M} = \bar{u}(p')(-ie\gamma^\mu)\,\varepsilon^*_\mu(k')\,\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}\,(-ie\gamma^\nu)\,\varepsilon_\nu(k)\,u(p)$$

$$+\ \bar{u}(p')(-ie\gamma^\nu)\,\varepsilon_\nu(k)\,\frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2}\,(-ie\gamma^\mu)\,\varepsilon^*_\mu(k')\,u(p)$$

$$= -ie^2\,\varepsilon^*_\mu(k')\,\varepsilon_\nu(k)\,\bar{u}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2}+\frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u(p)$$

The denominators

$$(p+k)^2-m^2 = 2p\cdot k \qquad (p-k')^2-m^2 = -2p\cdot k'$$

The numerators

$$(\not{p}+m)\gamma^\nu u(p) = (2p^\nu - \gamma^\nu\not{p} + \gamma^\nu m)\,u(p)$$

$$= 2p^\nu u(p) - \gamma^\nu(\not{p}-m)\,u(p)$$

$$= 2p^\nu u(p)$$

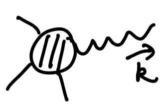
Then

$$i\mathcal{M} = -ie^2\,\varepsilon^*_\mu(k')\,\varepsilon_\nu(k)\,\bar{u}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu+2\gamma^\mu p^\nu}{2p\cdot k}+\frac{-\gamma^\nu\not{k}'\gamma^\mu+2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p)$$

The sum of photon polarization vectors :

$$\sum_{pol}\varepsilon^*_\mu\,\varepsilon_\nu \longrightarrow -g_{\mu\nu} \qquad \text{[valid in a QED amplitude]}$$

Consider an arbitary QED processes involving an exteral photon with k^μ



$= i\mathcal{M}^\mu(k)\,\varepsilon^*_\mu(k,\lambda)$

Then

$$\sum_\lambda |\varepsilon^*_\mu(k,\lambda)\,\mathcal{M}^\mu(k)|^2 = \sum_\lambda \varepsilon^*_\mu(k,\lambda)\,\varepsilon_\nu(k,\lambda)\,\mathcal{M}^\mu(k)\,\mathcal{M}^{\nu*}(k)$$

For simplicity $\quad k^\mu = (k, 0, 0, k)$

$$\varepsilon^\mu(k,1) = (0, 1, 0, 0) \qquad \varepsilon^\nu(k,2) = (0, 0, 1, 0)$$

$$\Rightarrow \sum_{\lambda=1}^{2} |\varepsilon^*_\mu(k,\lambda)\,\mathcal{M}^\mu(k)|^2 = |\mathcal{M}^1(k)|^2 + |\mathcal{M}^2(k)|^2$$

can be violated by

The matrix element $\quad \mathcal{M}^\mu(k) = \int d^4x\, e^{ik\cdot x} \langle f|\, j^\mu(x)\,|i\rangle$

Since the current conservation $\quad \partial_\mu j^\mu = 0 \quad \Rightarrow \quad k_\mu \mathcal{M}^\mu = 0$.

$\mathcal{M} = \mathcal{M}^\mu \varepsilon^*_\mu(k)$ vanishes as $\quad \varepsilon^*_\mu(k) \to k_\mu \qquad$ " Ward identity "

In our case, $\quad k^\mu \mathcal{M}_\mu = 0 \Leftrightarrow k\,\mathcal{M}^0(k) - k\,\mathcal{M}^3(k) = 0. \quad \Rightarrow \mathcal{M}^0 = \mathcal{M}^3$

Then

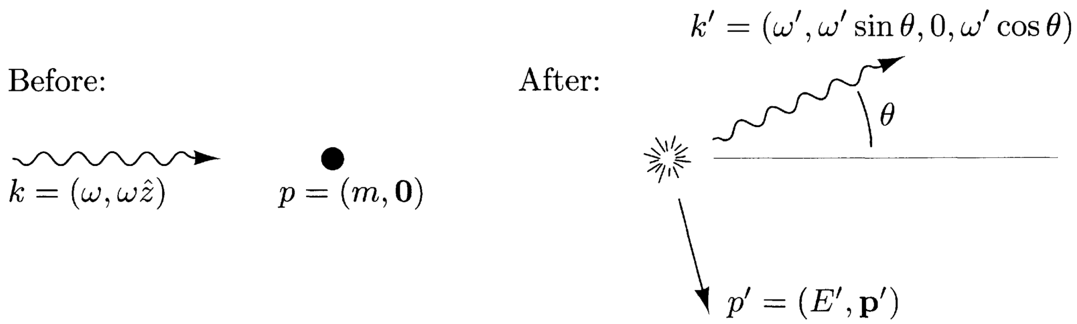
$$\begin{aligned}\sum_\lambda \varepsilon^*_\mu(k,\lambda)\,\varepsilon_\nu(k,\lambda)\,\mathcal{M}^\mu(k)\,\mathcal{M}^{\nu*}(k) &= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 \\ &= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2 - |\mathcal{M}^0|^2 \\ &= -g_{\mu\nu}\,\mathcal{M}^\mu(k)\,\mathcal{M}^{\nu*}(k)\end{aligned}$$

The unphysical time-like and longitudial photons can be omitted.

So for the Compton scattering

$$\frac{1}{4}\sum_{\text{spin}} |\mathcal{M}|^2 = 2e^4 \left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2 \right]$$

In the lab frame :



The differential cross section: [Klein - Nishina formula 1929]

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right] \qquad \frac{\omega}{\omega'} = 1 + \frac{\omega}{m}(1-\cos\theta)$$

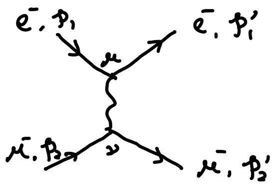
In the limit $\omega \to 0$, $\omega'/\omega \to 1$ $\qquad \sigma = \frac{8\pi\alpha^2}{3m^2} \sim \pi r_e^2$ $\qquad [\, r_e = \frac{\alpha}{m} \,]$

classical electron radius

J. J. Thomson

6.3 Crossing Symmetry [Peskin 5.4]

Electron - Muon Scattering $e^-\mu^- \to e^-\mu^-$



$$= \frac{ie^2}{q^2}\, \bar{u}(p_1')\gamma^\mu u(p_1)\, \bar{u}(p_2')\gamma_\mu u(p_2)$$

Then the squared amplitude, averaged and summed over spin

$$\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{e^4}{4q^2}\, \mathrm{Tr}\left[(\not{p}_1' + m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right]\, \mathrm{Tr}\left[(\not{p}_2' + m_\mu)\gamma_\mu(\not{p}_2 + m_\mu)\gamma_\nu\right]$$

This is exactly the same as the result for $e^+e^- \to \mu^+\mu^-$, with the replacement.

$$p \to p_1 \,,\quad p' \to -p_1' \,,\quad k \to p_2' \quad k' \to -p_2$$

In the high energy limit $E \gg m_\mu , m_e$ the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2E_{cm}^2(1-\cos\theta)^2}\left[\, 4 + (1+\cos\theta)^2 \,\right]$$

$$\propto \frac{1}{\theta^4} \qquad \text{as } \theta \to 0$$

Such singularity arises from the nearly on-shell virtual photon ($q^2 \approx 0$).

$$\boxed{\text{Crossing Symmetry}}$$

$$\mathcal{M}(\,\phi(p) + \cdots \to \cdots\,) = \mathcal{M}(\,\cdots \to \cdots\, \bar{\phi}(k)\,)$$

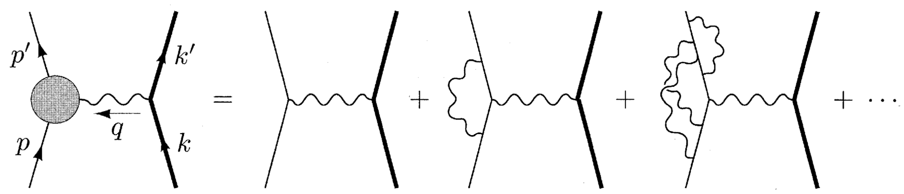
antiparticle of ϕ with $k = -p$

Either amplitude can be obtained from the other by analytic continuation.

For each crossed fermion one needs to cancel one minus sign by hand.

6.4 The electron vertex function: [Peskin 6.2 & 6.3]

We consider the correction to e^- scattering from an virtual photon



$$i\mathcal{M} = ie^2 \left[\bar{u}(p') \Gamma^\mu(p', p)\, u(p) \right] \frac{1}{q^2} \left[\bar{u}(k') \gamma_\mu u(k) \right]$$

Let's consider a classical potential $A^{cl}_\mu(x)$ without quantizing the EM field $A_\mu(x)$. The interaction Hamiltonian

$$H_I = \int d^3x \; e \bar{\psi} \gamma^\mu \psi \, A^{cl}_\mu$$

If $A^{cl}_\mu(0, \vec{x})$ is time independent.

$$\langle p' | \, iT \, | p \rangle \equiv i\mathcal{M} \, (2\pi) \delta(E' - E) = -ie \, \bar{u}(p') \gamma^\mu u(p) \, \tilde{A}^{cl}_\mu(p' - p)$$

↑ Fourier transformation of $A^{cl}_\mu(x)$

If we consider the vertex corrections

$$i\mathcal{M} \, (2\pi) \delta(E' - E) = -ie \, \bar{u}(p') \Gamma^\mu(p', p) \, u(p) \, \tilde{A}^{cl}_\mu(p' - p)$$

In general, Γ^μ is some expression involving $p^\mu, p'^\mu, \gamma^\mu$ and m, e.

[QED : parity conserving theory]

Then

$$\Gamma^\mu = \gamma^\mu A + (p'^\mu + p^\mu) B + (p'^\mu - p^\mu) C$$

A, B, C must be functions of m, e, q^2

By applying Ward identity $q_\mu \Gamma^\mu = 0$ [do not require $q^2 = 0$]

$$q_\mu \Gamma^\mu = \underbrace{\not{q} A}_{\text{vanish}} + \underbrace{(q \cdot p' + q \cdot p) B}_{\text{vanish}} + q^2 C = 0$$

vanish, since $\bar{u}(p') \not{q} \, u(p)$

$= \bar{u}(p') (\not{p}' - \not{p}) \, u(p)$

$= \bar{u}(p') (m - m) \, u(p) = 0$

vanish $q \cdot (p' + p) = (p')^2 - p^2 = 0$

So $C = 0$, only A, B survive.

By applying the Gorden identity :

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \qquad \left(\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\right)$$

We have

$$\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)$$

F_1 and F_2 are called form factors. To lowest order $F_1 = 1$, $F_2 = 0$

$\boxed{\text{the electric charge :}}$

We compute the elastic scattering of a nonrelativistic electron from a region of electrostatic potential : $A_\mu^{cl}(x) = (\phi(\vec{x}), \vec{0}) \Rightarrow \tilde{A}_\mu^{cl}(q) = (2\pi\delta(q^0)\tilde{\phi}(\vec{q}), \vec{0})$

Then

$$i\mathcal{M} = -ie\,\bar{u}(p')\Gamma^0(p', p)u(p)\,\tilde{\phi}(\vec{q})$$

If the electrostatic potential is slowly varying , then $q \to 0$.

In the nonrelativistic limit ($E \gg \vec{p}$)

$$u^s(p) = \sqrt{m}\begin{pmatrix}\xi_s \\ \xi_s\end{pmatrix} + O(|\vec{p}|)$$

then

$$\bar{u}(p')\gamma^0 u(p) = u^\dagger(p')(\gamma^0)^2 u(p) = 2m\,\xi'^\dagger\xi$$

the amplitude

$$i\mathcal{M} = -ie\,F_1(0)\,2m\,\xi'^\dagger\xi$$

This is the Born appoximation for scattering from a potential.

$$V(\vec{x}) = e\,F_1(0)\,\phi(\vec{x})$$

$F_1(0)$ is the electric charge of the electron . Since $F_1(0) = 1$ at the LO, radiative corrections should vanish at $q^2 = 0$.

$\boxed{\text{the electric magnetic moment.}}$

Similarly, considering a static vector potential. $A_\mu^{cl}(x) = (0, \vec{A}^{cl}(\vec{x}))$

Then the amplitude

$$i\mathcal{M} = +ie\left[\bar{u}(p')\left(\gamma^i F_1 + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2\right)u(p)\right]\tilde{A}^i_{cl}(\vec{q})$$

Since the Fourier transform of magnetic field is

$$\tilde{B}^k(\vec{q}) = -i\,\varepsilon^{ijk}q^i\,\tilde{A}^j_{cl}(\vec{q})$$

We need carefully extract a contribution linear in q^i

The nonrelativistic expansion of the spinors $u(p)$

$$u(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi\\ \sqrt{p\cdot\bar{\sigma}}\,\xi\end{pmatrix} = \sqrt{m}\begin{pmatrix}(1-\vec{p}\cdot\vec{\sigma}/2m)\,\xi\\ (1+\vec{p}\cdot\vec{\sigma}/2m)\,\xi\end{pmatrix} + O(p^2)$$

Then the F_1 term

$$\begin{aligned}
\bar{u}(p')\gamma^i u(p) &= u^\dagger(p')\gamma^0\gamma^i u(p)\\
&= u^\dagger(p')\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}u(p)\\
&= u^\dagger(p')\begin{pmatrix}-\sigma^i&0\\0&\sigma^i\end{pmatrix}u(p)\\
&= m\left(\xi^\dagger\left(1-\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\quad \xi^\dagger\left(1+\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\right)\begin{pmatrix}-\sigma^i&0\\0&\sigma^i\end{pmatrix}\begin{pmatrix}(1-\vec{p}\cdot\vec{\sigma}/2m)\xi\\(1+\vec{p}\cdot\vec{\sigma}/2m)\xi\end{pmatrix}\\
&= 2m\,\xi^\dagger\left(\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\sigma^i + \sigma^i\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi
\end{aligned}$$

$$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k \rightarrow \quad = \xi^\dagger\left[p'^j(\delta^{ij} + i\varepsilon^{jik}\sigma^k) + (\delta^{ij} + i\varepsilon^{ijk}\sigma^k)p^j\right]\xi$$

$$= \xi^\dagger\left[\underbrace{p'^i + p^j}_{\text{spin independent}}\ \underbrace{-\,i\varepsilon^{ijk}(p'^j - p^j)\sigma^k}_{\text{spin dependent terms}}\right]\xi$$

magnetic moment interaction

$$\Rightarrow \quad 2m\,\xi^\dagger\left(\frac{-i}{2m}\varepsilon^{ijk}q^j\sigma^k\right)\xi$$

For the F_2 term

$$\bar{u}(p')\frac{i}{2m}\sigma^{i\nu}q_\nu u(p) = 2m\,\xi'^\dagger\left(\frac{-i}{2m}\varepsilon^{ijk}q^j\sigma^k\right)\xi.$$

Then the complete term linear in q is

$$2m\,\xi'^{\dagger}\left(\frac{-i}{2m}\,\varepsilon^{ijk} q^{j}\sigma^{k}\,[\,F_1(0)+F_2(0)\,]\right)\xi$$

the amplitude

$$i\mathcal{M} = -i\,(2m)\,e\,\xi'^{\dagger}\left[\frac{-1}{2m}\,\sigma^{k}\,(\,F_1(0)+F_2(0)\,)\right]\xi\,\tilde{B}^{k}(\vec{q})$$

Then the potential

$$V(\vec{x}) = -\vec{\mu}\cdot\vec{B}(\vec{x})$$

$\uparrow$ magnetic moment $\quad \vec{\mu} = \frac{e}{m}[\,F_1(0)+F_2(0)\,]\,\xi'^{\dagger}\frac{\vec{\sigma}}{2}\,\xi$

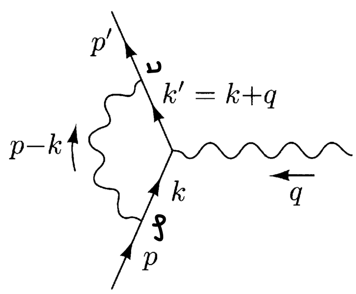
In the standard form : $\vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S}$ $\quad \uparrow$ electron spin

$\uparrow$ Landé g-factor

$$g = 2\,[\,F_1(0)+F_2(0)\,] = 2 + 2F_2(0)$$

At the LO. $g=2$, QED predicts $g = 2 + O(\alpha)$ anomalous magnetic moment

$$\boxed{\text{One-loop calculation to the electron vertex function}}$$



To order α, $\Gamma^{\mu} = \gamma^{\mu} + \delta\Gamma^{\mu}$

$$\bar{u}(p')\,\delta\Gamma^{\mu}(p',p)\,u(p) = \int \frac{d^4k}{(2\pi)^4}\,\frac{-i g_{\nu\rho}}{(k-p)^2+i\varepsilon}\,\bar{u}(p')(-ie\gamma^{\nu})\frac{i(\not{k}'+m)}{k'^2-m^2+i\varepsilon}\,\gamma^{\mu}\,\frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}(-ie\gamma^{\rho})u(p)$$

$$\gamma^{\nu}\gamma^{\mu}\gamma^{\nu} = -2\gamma^{\mu} \dashrightarrow \; = 2ie^2\int\frac{d^4k}{(2\pi)^4}\,\frac{\bar{u}(p')[\,\not{k}\gamma^{\mu}\not{k}' + m^2\gamma^{\mu} - 2m(k+k')^{\mu}\,]u(p)}{[\,(k-p)^2+i\varepsilon\,]\,[\,k'^2-m^2+i\varepsilon\,]\,[\,k^2-m^2+i\varepsilon\,]} \qquad (*)$$

Here $i\varepsilon$ terms cannot be dropped in the loop momentum integral.

$\boxed{\text{Feynman parameters}}$

$$\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \quad \delta(x+y-1) \frac{1}{[xA+yB]^2} \qquad [\, x, y : \text{ Feynman parameters} \,]$$

e.g.

$$\frac{1}{(k-p)^2(k^2-m^2)} = \int_0^1 dx\, dy\; \delta(x+y-1) \frac{1}{[x(k-p)^2 + y(k^2-m^2)]^2}$$

$$= \int_0^1 dx\, dy\; \delta(x+y-1) \frac{1}{[k^2 - 2xk\cdot p + xp^2 - ym^2]^2}$$

let $\quad l \equiv k - xp \quad$ then $\quad d^4k = d^4 l$

$$k^2 - 2xk\cdot p + xp^2 - y^2m^2$$
$$= (k-xp)^2 - x^2p^2 + xp^2 - y^2m^2$$
$$= l^2 - x^2p^2 + xp^2 - y^2m^2$$

$$\frac{1}{A_1A_2\cdots A_n} = \int_0^1 dx_1\cdots dx_n\, \delta(\textstyle\sum x_i - 1) \frac{(n-1)!}{\left[x_1A_1 + x_2A_2 + \cdots x_nA_n\right]^n}. \quad (6.41)$$

The proof of this identity is by induction. The case $n = 2$ is just Eq. (6.39); the induction step is not difficult and involves the use of (6.40).

By repeated differentiation of (6.41), you can derive the even more general identity

$$\frac{1}{A_1^{m_1}A_2^{m_2}\cdots A_n^{m_n}} = \int_0^1 dx_1\cdots dx_n\, \delta(\textstyle\sum x_i - 1) \frac{\prod x_i^{m_i-1}}{\left[\sum x_iA_i\right]^{\Sigma m_i}} \frac{\Gamma(m_1+\cdots+m_n)}{\Gamma(m_1)\cdots\Gamma(m_n)}. \quad (6.42)$$

This formula is true even when the m_i are not integers; in Section 10.5 we will apply it in such a case.

Then the denominator in (*)

$$\frac{1}{[(k-p)^2+i\varepsilon][k'^2-m^2+i\varepsilon][k^2-m^2+i\varepsilon]} = \int_0^1 dx\, dy\, dz\; \delta(x+y+z-1) \frac{2}{D^3}$$

the denominator D is

$$D = x(k^2-m^2+i\varepsilon) + y(k'^2-m^2+i\varepsilon) + z[(k-p)^2+i\varepsilon]$$

$$\overset{?}{=} k^2 + 2k\cdot(yq - zp) + yq^2 + zp^2 - (x+y)m^2 + i\varepsilon$$

$x+y+z = 1, \quad k' = k+q$

let $\quad l \equiv k + yq - zp$

then $\quad D = l^2 - \Delta + i\varepsilon \qquad \Delta \equiv -xyq^2 + (1-z)^2 m^2$

$$q^2 < 0 \Rightarrow \Delta > 0$$

For the numerator

$$\int \frac{d^4 l}{(2\pi)^4} \frac{l^\mu}{D^3} = 0 \qquad \int \frac{d^4 l}{(2\pi)^4} \frac{l^\mu l^\nu}{D^3} = \int \frac{d^4 l}{(2\pi)^4} \frac{\frac{1}{4} g^{\mu\nu} l^2}{D^3}$$

odd function of l

if $\mu \neq \nu$ vanishes

To check : $g_{\mu\nu} l^\mu l^\nu = \frac{1}{4} g_{\mu\nu} g^{\mu\nu} l^2$

$$l^2 = l^2$$

$$\text{Numerator} = \bar{u}(p') [\, \not{k} \gamma^\mu \not{k}' + m^2 \gamma^\mu - 2m(k+k')^\mu \,] u(p)$$

$$\to \bar{u}(p') [-\tfrac{1}{2} \gamma^\mu l^2 + \cdots]$$

$$= \bar{u}(p') [\, \gamma^\mu (-\tfrac{1}{2} l^2 + (1-x)(1-y) q^2 + (1-2z-z^2) m^2) + (p^\mu + p'^\mu) m z (z-1)$$

$$+ q^\mu m (z-2)(x-y) \,] u(p)$$

vanishes [Ward identity]
odd $x \leftrightarrow y$

Using the Gordon identity

$$\bar{u}(p') \delta\Gamma^\mu(p',p) u(p) = 2ie^2 \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{2}{D^3}$$

$$\times \bar{u}(p') [\, \gamma^\mu \cdot (-\tfrac{1}{2} l^2 + (1-x)(1-y) q^2 + (1-4z+z^2) m^2$$

$$+ \frac{i\sigma^{\mu\nu} q_\nu}{2m} (2m^2 z (1-z)) \,] u(p)$$

with $\; D = l^2 - \Delta + i\varepsilon \;; \qquad \Delta = -xyq^2 + (1-z)^2 m^2 > 0$

Wick rotation

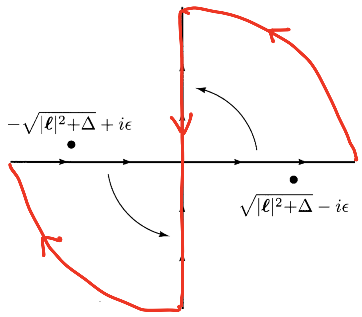


Figure 6.1. The contour of the ℓ^0 integration can be rotated as shown.

Define a Euclidean 4-momentum variable ℓ_E :

$$\ell^0 \equiv i\,\ell_E^0 \quad ; \qquad \vec{\ell} = \vec{\ell}_E$$

Then we can evaluate the integral in 4-dim spherical coordinates

E.g. $$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2-\Delta]^m} = \frac{i}{(-1)^m} \frac{1}{(2\pi)^4} \int d^4\ell_E \frac{1}{[\ell_E^2+\Delta]^m}$$

$$= \frac{i(-1)^m}{(2\pi)^4} \int d\Omega_4 \int_0^{+\infty} d\ell_E \frac{\ell_E^3}{[\ell_E^2+\Delta]^m}$$

$$\int d\Omega_4 = \int \sin^2 w \sin\theta \, d\phi \, d\theta \, dw = 2\pi^2$$

$$\ell_E^\mu = \ell_E(\sin w \sin\theta \cos\phi, \sin w \sin\theta \sin\phi, \sin w \cos\theta, \cos w)$$

Then we have

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2-\Delta]^m} = \frac{i(-1)^m}{(4\pi)^2} \frac{1}{(m-1)(m-2)} \frac{1}{\Delta^{m-2}}$$

Then the form factor $F_2(q^2)$

$$F_2(q^2) = \frac{\alpha}{2\pi} \int dx\,dy\,dz\,\delta(x+y+z-1) \left[\frac{2m^2 z(1-z)}{m^2(1-z)^2-q^2xy}\right] + O(\alpha^2)$$

$$\Rightarrow F_2(q^2=0) = \frac{\alpha}{\pi} \int_0^1 dz \int_0^{1-z} dy \frac{z}{1-z} = \frac{\alpha}{2\pi} \approx 0.0011614 \qquad \text{[Schwinger 1948]}$$

$$a_e^{Exp} = 0.0011597$$

6.5 Field - Strength Renormalization [Peskin 7.1] " Non-perturbative "

We will investigate the analytic structure of the two point correlation function.

$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$

try to insert a complete set of states.

eigenstates of the full interacting Hamiltonian, H or momentum $\vec{P}$

$[H, \vec{P}] = 0$

Let $|\lambda_0\rangle$ be an eigenstate of H with momentum zero $\vec{P}|\lambda_0\rangle = \vec{0}$

Then any eigenstate of H can be written as the boost of $|\lambda_0\rangle$

The eigenvalue of the 4-momentum operator $P^\mu = (H, \vec{P})$

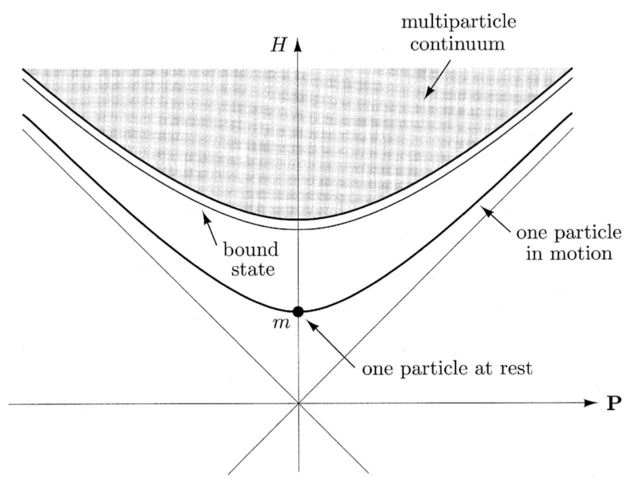


Figure 7.1. The eigenvalues of the 4-momentum operator $P^\mu = (H, \mathbf{P})$ occupy a set of hyperboloids in energy-momentum space. For a typical theory the states consist of one or more particles of mass m. Thus there is a hyperboloid of one-particle states and a continuum of hyperboloids of two-particle states, three-particle states, and so on. There may also be one or more bound-state hyperboloids below the threshold for creation of two free particles.

In the free theory, the completeness relation for one-particle state:

$$(1)_{1\text{-particle}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} |\vec{p}\rangle\langle\vec{p}|$$

Let $|\lambda_p\rangle$ be the boost of $|\lambda_0\rangle$ with $\vec{p}$ and $E_p(\lambda) \equiv \sqrt{|\vec{p}|^2 + m_\lambda^2}$ is the energy

Then the completeness relation is

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} |\lambda_p\rangle\langle\lambda_p|$$

sum over all $|\lambda_0\rangle$

Assume that $x^0 > y^0$, insert $\mathbb{1}$ into two-point function

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} \langle\Omega|\phi(x)|\lambda_p\rangle\langle\lambda_p|\phi(y)|\Omega\rangle \qquad [\text{ drop } \langle\Omega|\phi|\Omega\rangle \ ]$$

Then the matrix element

$$\langle\Omega|\phi(x)|\lambda_p\rangle = \langle\Omega| e^{iP\cdot x} \phi(0) e^{-iP\cdot x} |\lambda_p\rangle$$

$$= \langle\Omega|\phi(0)|\lambda_p\rangle e^{-ip\cdot x}\Big|_{p^0=E_p}$$ $\Leftarrow$. assume vaccum is invariant under Lorentz transformation and translation

$$= \langle\Omega| U^{-1} U \phi(0) U^{-1} U |\lambda_p\rangle e^{-ip\cdot x}\Big|_{p^0=E_p}$$

Lorentz boost from $\vec{p}$ to $\vec{0}$

$$= \langle\Omega|\phi(0)|\lambda_0\rangle e^{-ip\cdot x}\Big|_{p^0=E_p} \qquad \Leftarrow \qquad U\phi(0)U^{-1} = \phi(0) \ ; \ U|\Omega\rangle = |\Omega\rangle$$

Then for $x^0 > y^0$

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_{\lambda} \int \frac{d^4p}{(2\pi)^3} \frac{i}{p^2 - m_\lambda^2 + i\varepsilon} |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 e^{-ip\cdot(x-y)}$$

Finally , the two point function

$$\langle\Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle = \int_0^{\infty} \frac{dM^2}{2\pi} \rho(M^2) D_F(x-y; M^2)$$ [Källén - Lehmann spectral representation]

where spectral density: $\rho(M^2) = \sum_{\lambda} (2\pi)\delta(M^2 - m_\lambda^2) |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$ [positive]

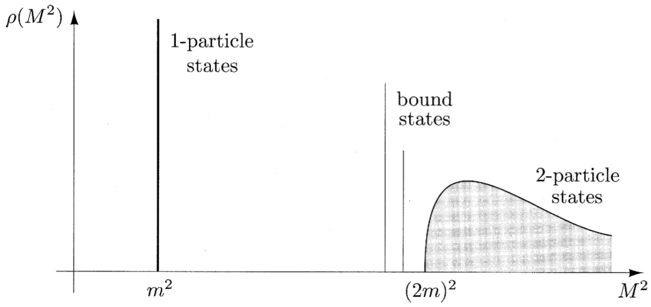


Figure 7.2. The spectral function $\rho(M^2)$ for a typical interacting field theory. The one-particle states contribute a delta function at m^2 (the square of the particle's mass). Multiparticle states have a continuous spectrum beginning at $(2m)^2$. There may also be bound states.

The one-particle state contributes an isolated delta function

$$\rho(M^2) = 2\pi\,\delta(M^2 - m^2)\cdot Z + \cdots$$

Z: field-strength renormalization

determined by $|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$ [the probability for $\phi(0)$ to create a state $|\lambda_0\rangle$ from $|\Omega\rangle$

$0 \le Z \le 1$

$|\lambda_0\rangle$: single particle state with 0- momentum

m is the exact mass of a single particle different from mass in $\mathcal{L}(m_0)$

m_0 : bare mass

m : physical mass [observable]

In the momentum space:

$$\int d^4x\, e^{ip\cdot x} \langle\Omega|T\phi(x)\phi(0)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon}$$

$$= \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim 4m^2}^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon}$$

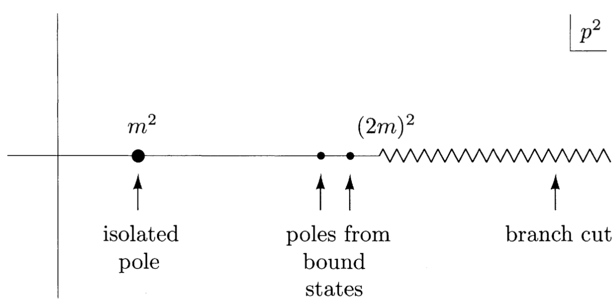


Figure 7.3. Analytic structure in the complex p^2-plane of the Fourier transform of the two-point function for a typical theory. The one-particle states contribute an isolated pole at the square of the particle mass. States of two or more free particles give a branch cut, while bound states give additional poles.

The two point function of Dirac fields.

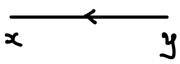
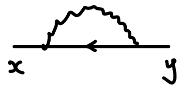
$$\int d^4x\, e^{ip\cdot x} \langle\Omega|T\psi(x)\bar\psi(0)|\Omega\rangle = \frac{iZ_2(\not p + m)}{p^2 - m^2 + i\epsilon} + \cdots$$

with $\langle\Omega|\psi(0)|p.s\rangle = \sqrt{Z_2}\, u^s(p)$

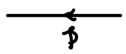
one particle eigenstate with $\vec p$, s

The electron self-energy

The electron two-point function :

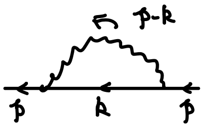
$\langle \Omega | T \psi(x) \bar{\psi}(y) | \Omega \rangle =$  $+$  $+ \cdots$

The first diagram is

 $= \dfrac{i(\not{p}+m_0)}{p^2 - m_0^2 + i\varepsilon}$

write m_0 instead of m , $m^2 = m_0^2 + o(\alpha)$

The second diagram



$= \dfrac{i(\not{p}+m_0)}{p^2 - m_0^2} \left[-i\Sigma_2(p) \right] \dfrac{i(\not{p}+m_0)}{p^2 - m_0^2}$

with $\quad -i\Sigma_2(p) = (-ie)^2 \displaystyle\int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k}+m_0)}{k^2 - m_0^2 + i\varepsilon} \gamma_\mu \frac{-i}{(p-k)^2 - \mu^2 + i\varepsilon}$

photon mass , regularize the IR div.

Introduce a Feynman parametr

$$\frac{1}{k^2 - m_0^2 + i\varepsilon} \frac{1}{(p-k)^2 - \mu^2 + i\varepsilon} = \int_0^1 dx \frac{1}{\left[k^2 - 2xk\cdot p + xp^2 - x\mu^2 - (1-x)m_0^2 + i\varepsilon \right]^2}$$

let $\ell \equiv k - xp$, then

$$-i\Sigma_2(p) = -e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{-2x\not{p} + 4m_0}{\left[\ell^2 - \Delta + i\varepsilon \right]^2} \qquad \left[\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2 \right]$$

Using Pauli - Villars procedure and replacing the photon propagator

$$\frac{1}{(p-k)^2 - \mu^2 + i\varepsilon} \longrightarrow \frac{1}{(p-k)^2 - \mu^2 + i\varepsilon} - \frac{1}{(p-k)^2 - \Lambda^2 + i\varepsilon}$$

Now Wick-rotate and substitute the Euclidean variable $\ell_E^0 = -i\ell^0$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\left[\ell^2 - \Delta \right]^2} \longrightarrow \frac{i}{(4\pi)^2} \int_0^\infty d\ell_E^2 \left(\frac{\ell_E^2}{\left[\ell_E^2 + \Delta \right]^2} - \frac{\ell_E^2}{\left[\ell_E^2 + \Delta_\Lambda \right]^2} \right)$$

$$= \frac{i}{(4\pi)^2} \log \frac{\Delta_\Lambda}{\Delta}$$

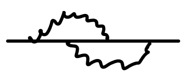
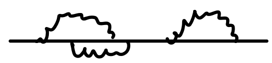
with $\Delta_\Lambda = -x(1-x)p^2 + x\Lambda^2 + (1-x)m_0^2 \xrightarrow{\Lambda \to \infty} x\Lambda^2$

The final result is

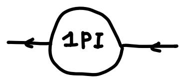
$$\Sigma_2(p) = \frac{\alpha}{2\pi}\int_0^1 dx\ (2m_0 - x\not{p})\log\left[\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right]$$

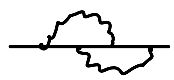
To find the simple at $p^2 = m^2$

define a one-particle irreducible (1PI) diagram

 is 1PI ;  is not 1PI

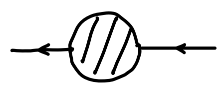
define $-i\Sigma(p)$ as the sum of all 1PI

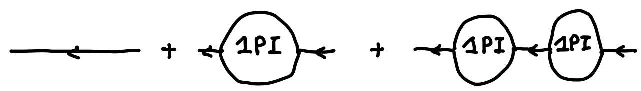
$-i\Sigma(p) =$ 

$=$  $+$  $+\cdots$

to leading order in α we see that $\Sigma = \Sigma_2$

Then the two-point function

$\int d^4x\ \langle\Omega|\, T\,\psi(x)\bar{\psi}(0)\,|\Omega\rangle\, e^{ip\cdot x} =$ 

$=$ 

$$= \frac{i(\not{p}+m_0)}{p^2-m_0^2} + \frac{i(\not{p}+m_0)}{p^2-m_0^2}(-i\Sigma)\frac{i(\not{p}+m_0)}{p^2-m_0^2} + \cdots$$

$$= \frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}\frac{\Sigma(\not{p})}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}\left(\frac{\Sigma(\not{p})}{\not{p}-m_0}\right)^2$$

$$= \frac{i}{\not{p}-m_0-\Sigma(\not{p})}$$

Then the physical mass m is the solution of

$$[\not{p} - m_0 - \Sigma(\not{p})]\Big|_{\not{p}=m} = 0$$

Close to the pole, the denominator has the form of

$$(\not{p}-m)\cdot\left(1 - \frac{d\Sigma}{d\not{p}}\Big|_{\not{p}=m}\right) + O((\not{p}-m)^2)$$

Then the full electron propagator has a single-particle pole with

$$Z_2^{-1} = 1 - \frac{d\Sigma}{d\not{p}}\Big|_{\not{p}=m}$$

Then $\quad \delta m = m - m_0 = \Sigma_2(\not{p} = m) \approx \Sigma_2(\not{p} = m_0)$

Thus

$$\delta m = \frac{\alpha}{2\pi} m_0 \int_0^1 dx \; (2-x) \log \frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}$$

$$\overset{\Lambda\to\infty}{\sim} \quad \frac{3\alpha}{4\pi} m_0 \log \frac{\Lambda^2}{m_0^2} \qquad [\text{ log divergent }]$$

The validity of perturbation theory would be more plausible if we could compute Feynman diagrams using $\frac{i}{\not{p}-m}$ instead of $\frac{i}{\not{p}-m_0}$

LSZ reduction formula :

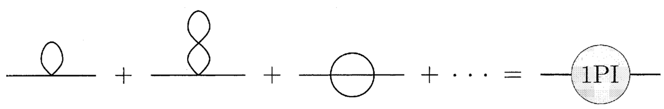
$$\prod_1^n \int d^4x_i\, e^{ip_i\cdot x_i} \prod_1^m \int d^4y_j\, e^{-ik_j\cdot y_j}\ \langle\Omega| T\{\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)\}|\Omega\rangle$$
$$\underset{\substack{\text{each } p_i^0\to+E_{\mathbf{p}_i}\\ \text{each } k_j^0\to+E_{\mathbf{k}_j}}}{\sim} \left(\prod_{i=1}^n \frac{\sqrt{Z}\, i}{p_i^2-m^2+i\epsilon}\right)\left(\prod_{j=1}^m \frac{\sqrt{Z}\, i}{k_j^2-m^2+i\epsilon}\right)\langle \mathbf{p}_1\cdots\mathbf{p}_n|S|\mathbf{k}_1\cdots\mathbf{k}_m\rangle . \tag{7.42}$$

E.g. the scalar field four-point function in $\lambda\phi^4$ theory.

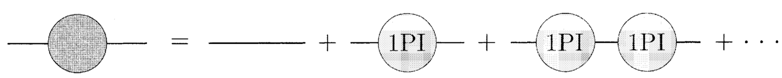
The exact four-point function

$$\left(\prod_1^2 \int d^4x_i\, e^{ip_i\cdot x_i}\right)\left(\prod_1^2 \int d^4y_i\, e^{-ik_i\cdot y_i}\right)\langle\Omega|T\{\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\}|\Omega\rangle$$

Sum up the corrections to each external leg

$-iM^2(p^2) =$ 

The full propagator is :



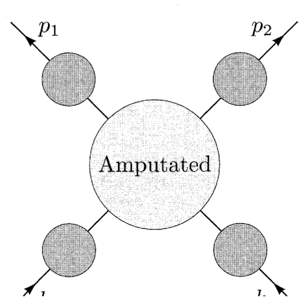
$$= \frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}(-iM^2)\frac{i}{p^2-m_0^2} + \cdots$$
$$= \frac{i}{p^2-m_0^2-M^2(p^2)}. \tag{7.43}$$

Expand resummed propagator about the physical particle pole

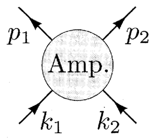
$$\frac{i}{p^2-m_0^2-M^2} \overset{p^0\to E_p}{\sim} \frac{iZ}{p^2-m^2} + (\text{regular})$$

Then for four legs

$$\frac{iZ}{p_1^2-m^2}\frac{iZ}{p_2^2-m^2}\frac{iZ}{k_1^2-m^2}\frac{iZ}{k_2^2-m^2}.$$



Finally.

$$\langle \mathbf{p}_1 \mathbf{p}_2 | \, S \, | \mathbf{k}_1 \mathbf{k}_2 \rangle = \left(\sqrt{Z}\right)^4$$ 

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