4.4 Renormalization of QED

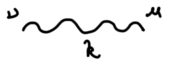
QED is completely specified by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 + \bar{\psi} (i \gamma^\mu D_\mu - m) \psi \qquad \text{" } D_\mu = \partial_\mu + ie A_\mu \text{ "}$$

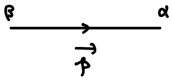
We here implement the gauge-fixing directly in the Lagrangian. ξ is an arbitrary gauge-fixing parameter.

Feynman rules of QED

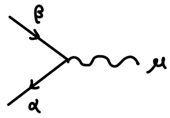
Propagators:



$$\frac{i}{k^2 + i\varepsilon} \left[-g_{\mu\nu} + (1-\xi) \frac{k_\mu k_\nu}{k^2} \right] \qquad \xi = 1 \text{ is Feynman gauge}$$

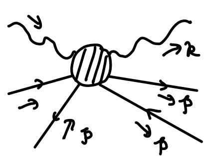


$$\frac{i(\not{p} + m)_{\alpha\beta}}{p^2 - m^2 + i\varepsilon} \qquad \text{" matrix } \alpha\beta \text{ in Dirac spinor space "}$$



$$-ie\,\gamma^\mu_{\alpha\beta}$$

incoming			outgoing	
photon	$\varepsilon_\mu(k,\lambda)$		$\varepsilon^*_\mu(k,\lambda)$	photon
electron	$u(p,s)$		$\bar{u}(p,s)$	electron
positron	$\bar{v}(p,s)$		$v(p,s)$	positron



Since the electron-photon interaction operator $\bar{\psi} \not{A} \psi$ has dimension four and $\Delta_i = 0$, QED is renormalizable by power countering.

Start from $\mathcal{L}$ with bare quantities and introduce renormalized parameters and fields

$$\psi_0(x) = \sqrt{Z_2}\, \psi(x) \qquad A_0^\mu(x) = \sqrt{Z_3}\, A^\mu(x)$$

$$m_0 = Z_m m \qquad e_0 = Z_e e$$

$$\xi_0 = Z_\xi \xi$$

Then

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + \bar{\psi}(i\not{\partial} - m - e\not{A})\psi \\
& -\frac{1}{4}(Z_3 - 1) F_{\mu\nu} F^{\mu\nu} - \left(\frac{Z_3}{Z_\xi} - 1\right) \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \\
& + (Z_2 - 1) \bar{\psi} i\not{\partial} \psi - (Z_2 Z_m - 1) m \bar{\psi}\psi - (Z_e Z_2 Z_3^{1/2} - 1) e \bar{\psi} \not{A} \psi \\
= & \ \mathcal{L}_r + \mathcal{L}_{ct}
\end{aligned}$$

gives rise to counterterm vertices

- Will show / discuss later that gauge symmetry relates the renormalization constants of terms containing A^μ:

$Z_\xi = Z_3$ $\Downarrow$ no counterterm vertex for $(\partial_\mu A^\mu)^2$

$Z_e = 1/\sqrt{Z_3}$

charge renormalization related to photon field renormalization

- The renormalized Lagrangian does not contain all possible terms of dimension less than four.

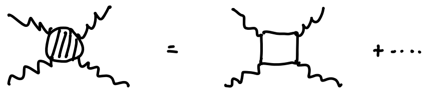
$A_\mu A^\mu$

photon mass term

$\underbrace{(\partial_\mu A_\nu) A^\mu A^\nu \qquad A^\mu A^\nu A_\mu A_\nu}_{\text{photon self interaction}}$

These terms are forbidden by gauge symmetry (provided the regularization preserve gauge symmetry)

This implies that the photon 4-pt fn



is finite and can be computed, once the mass and charge counterterms are determined from other measurements.

On-shell renormalization

Can define all renormalized fields & parameters in the $\overline{MS}$ scheme. This would be suitable for high-energy processes $\sqrt{s} \gg m_e$.

Another conventional scheme is the on-shell scheme.

Z_2, Z_m are determined from the electron 2-pt fn

$$S_0^{(2)}(p) = \int d^4x\, e^{ip\cdot x} \langle \Omega | T\{ \psi_0(x) \bar{\psi}_0(0) \} | \Omega \rangle$$

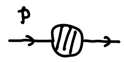
bare

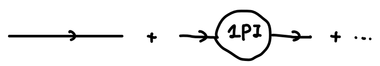
$$\xrightarrow[p^2 \to m^2]{} Z_2 \frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon} + \text{non-singular.}$$

on-shell field ren.

physical electron mass (511 keV)

In the renormalized perturbation theory.

$S^{(2)}(p) =$ 

$=$ 

$$= \frac{i}{\not{p} - m - \Sigma(p,m)} \xrightarrow[p^2 \to m^2]{} \frac{i}{\not{p} - m + i\varepsilon}$$

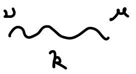
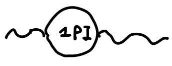
renormalized self-energy including counterterms

Then Z_2^{os} , Z_m^{os} determined from the renormalization conditions

$$\Sigma(p, m) = m\, \Sigma_1(p^2, m^2)\, \mathbb{1} + \Sigma_2(p^2, m^2)\, \not{p} \underset{p^2 \to m^2}{=} 0$$

$$\left.\frac{\partial \Sigma(p,m)}{\partial \not{p}}\right|_{p^2 = m^2} = 0$$

Z_3, Z_ξ are determined from the photon 2-pt fn.

$G^{(2)}(k) =$  $+$  $+ \cdots$

$$= \frac{i}{k^2 + i\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

$$+ \frac{i}{k^2+i\varepsilon}\left[-g_{\mu\rho} + (1-\xi)\frac{k_\mu k_\rho}{k^2}\right] i\,\Pi_{\rho\sigma}(k) \frac{i}{k^2+i\varepsilon}\left[-g_{\sigma\nu} + (1-\xi)\frac{k_\sigma k_\nu}{k^2}\right] + \cdots$$

renormalized photon self-energy
(including counterterms)

The most general form is

$$\Pi_{\mu\nu}(k) = (k^2 g_{\mu\nu} - k_\mu k_\nu)\,\Pi(k^2) + k_\mu k_\nu\, \Pi_2(k^2)$$

This vanishes at one loop. Will show for general gauge theories that this is true to any order : $\Pi_2(k^2) \equiv 0$

$$\Rightarrow \quad G^{(2)}(k) = \frac{i}{k^2+i\varepsilon}\left[\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1-\Pi(k^2)} - \xi\frac{k_\mu k_\nu}{k^2}\right]$$

$$\xrightarrow[k^2 \to 0]{} \quad \frac{i}{k^2+i\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right] + \text{non-singular}$$

on-shell scheme

hence Z_3^{os} is determined from $\Pi(0) = 0$

Note : The vanishing of Π_2 implies that the $\xi\frac{k_\mu k_\nu}{k^2}$ term is not modified and this implies $Z_\xi = Z_3$.

This is seen more clearly for the unrenormalized 2-pt fn.

$$G_0^{(2)}(k) = \frac{i}{k^2+i\varepsilon}\left[\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1-\Pi_0(k^2)} - \zeta_0 \frac{k_\mu k_\nu}{k^2}\right]$$

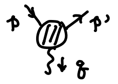
unrenormalized self-energy

$$\xrightarrow[k^2\to 0]{} \quad Z_3 \frac{i}{k^2+i\varepsilon}\left[-g_{\mu\nu} + (1-\zeta)\frac{k_\mu k_\nu}{k^2}\right] + \text{non-singular}$$

$$\Rightarrow \quad \frac{1}{Z_3} = 1 - \Pi_0(k^2=0) \qquad \text{and} \qquad Z_\zeta = Z_3$$

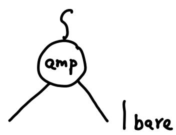
Z_e – renormalization of the strength of the interaction, i.e. the electric charge, is defined from the coupling of a photon to the electron at small momentum transfer $|\vec{q}| \ll m$.

$$\langle e^-(p',s')\,\gamma(q,\lambda);\,\text{out}\,|\,e^-(p,s);\,\text{in}\rangle \underset{q^\mu\to 0}{\equiv} (2\pi)^4\delta^{(4)}(p'+q-p) \qquad (*)$$
$$\times(-ie)\,\bar{u}(p',s')\gamma^\mu u(p,s)\,\varepsilon_\mu(q,\lambda)$$

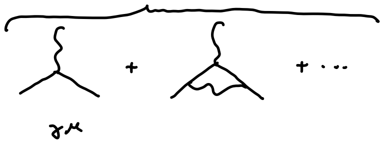
i.e. we require  is not corrected to any order for $q=0$.

This is similar to the definition for the strength of the 4-pt self interaction of the neutral scalar field.

To formulate the above condition more precisely, we analyze the amputated electron – photon 3-pt function:

$$\langle e^-(p',s')\,\gamma(q,\lambda);\,\text{out}\,|\,e^-(p,s);\,\text{in}\rangle = \sqrt{Z_3}\,Z_2 \times$$  $\times$ polarization fns

$$= (2\pi)^4\delta^{(4)}(p'+q-p)\sqrt{Z_3}\,Z_2\,(-ie_0)\,\bar{u}(p',s')\,\Gamma_0^\mu(p,p')\,u(p,s)\,\varepsilon_\mu^*(q,\lambda) \qquad (**)$$



$$\Gamma_0^\mu(p',p) = \gamma^\mu F_1^0(q^2,m) + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m} F_2^0(q^2,m)$$

Two scalar form factor (F_1 - Dirac , F_2 - magnetic)

The second structure vanishes for $q \to 0$, hence with $e_0 = Z_e e$ comparison of (*) , (**) leads to the on-shell charge renormalization condition.

$$\frac{1}{Z_e} = \sqrt{Z_3}\, Z_2\, F_1^0(q^2=0, m) \qquad \left[\begin{array}{l} F_1(q^2=0, m^2) = 1 \text{ for} \\ \text{the renormalized vertex fn.} \end{array} \right]$$

Ward identity:

$$F_1^0(q^2=0, m) = 1/Z_2$$

This implies $Z_e = 1/\sqrt{Z_3}$, i.e. a relation between charge and photon field renormalization.

Proof of the Ward identity: we use the Ward identity for the Noether current of the gauge symmetry inserted into the electron 2-pt function.

$$\partial_\mu^{(x)} \langle \Omega | T\{ j_0^\mu(x)\, \psi_{0\alpha}(z_1)\, \bar\psi_{0\beta}(z_2) \} | \Omega \rangle \qquad (*)$$

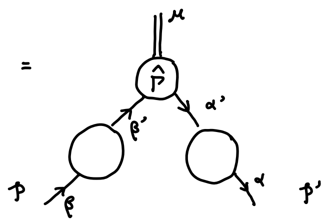
$$= \delta^{(4)}(x-z_1) \langle \Omega | T\{ \psi_{0\alpha}(z_1)\, \bar\psi_{0\beta}(z_2) \} | \Omega \rangle - \delta^{(4)}(x-z_2) \langle \Omega | T\{ \psi_{0\alpha}(z_1)\, \bar\psi_{0\beta}(z_2) \} | \Omega \rangle$$

where $j_0^\mu = -\bar\psi_0 \gamma^\mu \psi_0$ $\qquad F_\psi = i\psi \qquad [\, \delta\psi = \varepsilon F_\psi \,]$

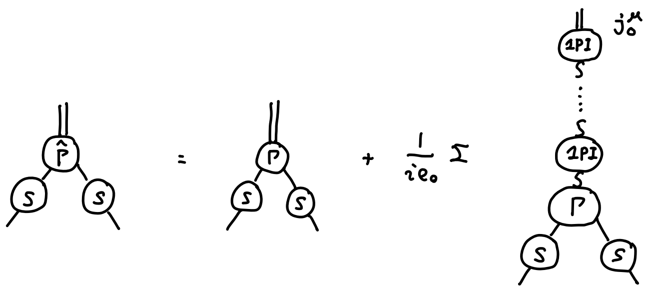
$\qquad F_{\bar\psi} = -i\bar\psi$

Take the Fourier transform $(-1)\int d^4x\, d^4z_1\, d^4z_2\, e^{iq\cdot x + ip'\cdot z_1 - ip\cdot z_2}$

LHS of (*) $= (2\pi)^4 \delta^{(4)}(q+p'-p)\,(-iq_\mu)\, S_{0\alpha\alpha'}(p')\, \hat\Gamma^\mu_{0\alpha'\beta'}(p',p)\, S_{0\beta'\beta}(p)$

$=$ 

Note : the relation between $\hat{\Gamma}$ and Γ is as follows :



However the extra contribution vanishes after multiplication with q_μ since

$$q_\mu \times \underbrace{\text{1PI}}_{i\Pi^{\mu\nu}(q)} = 0$$

so we may ignore the difference between Γ and $\hat{\Gamma}$ here.

RHS of $(*)$ $= (2\pi)^4 \delta^{(4)}(q + p' - q)(-1) \left[S_{0\alpha\beta}(p) - S_{0\alpha\beta}(p') \right]$

Multiply by the inverse of the fermion 2-pt fns :

$$(p' - p)_\mu \Gamma_0^\mu(p', p) = i S_0^{-1}(p') - i S_0^{-1}(p)$$

Now take the limit $q \to 0$, i.e. expand the RHS around p

$$\Rightarrow \quad \Gamma_0^\mu(p, p') \Big|_{p' \to p} = i \frac{\partial}{\partial p_\mu} S_0^{-1}(p)$$

Next we take the on-shell limit $p^2 \to m^2$ in which case $S_0(p) = \frac{i Z_2}{\not{p} - m + i\varepsilon}$ + non singular

Hence

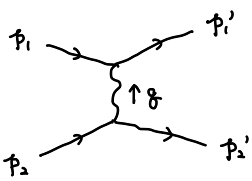
$$\gamma^\mu F_1^0(q^2 = 0, m^2) = i \frac{1}{i Z_2} \frac{\partial}{\partial p_\mu} (\not{p} - m) = \frac{1}{Z_2} \gamma^\mu$$

Scale dependence of the electromagnetic coupling

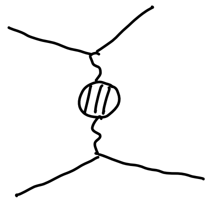
In the on-shell scheme the electromagnetic coupling is not scale-dependent.

Its value is $\alpha_{em} = \frac{e^2}{4\pi} = \frac{1}{137.036\ldots}$

The scale-dependence of the interaction strength enters through the photon vacuum polarization:



$\longrightarrow$ loop corrections



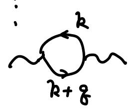
contains e^2/q^2

where $q = p_1' - p_1$

contains $\frac{e^2}{q^2} \times \frac{1}{1-\Pi(q^2)} \equiv \frac{e_{eff}^2(q)}{q^2}$

One-loop result :

$$i\Pi^{\mu\nu}(q^2) = (-1)(-ie)^2\,\tilde{\mu}^{2\varepsilon}\int\frac{d^dk}{(2\pi)^d}\,\frac{\mathrm{tr}[\gamma^\mu i(\not{k}+m)\gamma^\nu i(\not{k}+\not{q}+m)]}{(k^2-m^2+i\varepsilon)\,[(k+q)^2-m^2+i\varepsilon]}$$



$$+\ (-i)\,(Z_3-1)\,(q^2 g_{\mu\nu} - q_\mu q_\nu)$$

counterterm diagram 

$-\frac{1}{4}(Z_3-1)F_{\mu\nu}F^{\mu\nu}$ vertex

$$\Rightarrow\quad \Pi(q^2) = -\frac{2\alpha_{em}}{\pi}\,\Gamma(\varepsilon)\int_0^1 dx\ x\bar{x}\left(\frac{m^2 - x\bar{x}q^2 - i\varepsilon}{4\pi\tilde{\mu}^2}\right)^{-\varepsilon} - (Z_3-1)$$

Z_3 is determined in the on-shell scheme $\Pi(q^2=0)=0$. This gives

$$Z_3 = 1 - \frac{2\alpha_{em}}{\pi}\,\Gamma(\varepsilon)\int_0^1 dx\ x\bar{x}\left(\frac{m^2}{4\pi\tilde{\mu}^2}\right)^{-\varepsilon}$$

$$= 1 - \frac{\alpha_{em}}{3\pi}\left(\frac{1}{\varepsilon} - \ln\frac{m^2}{\mu^2}\right)$$

and $$\Pi(q^2) = \frac{2\alpha_{em}}{\pi}\int_0^1 dx\ x\bar{x}\,\ln\frac{m^2 - x\bar{x}q^2 - i\varepsilon}{m^2}$$

For the scattering of two electrons with three-momenta $\ll m$

$q^\mu = (q^0, \vec{q}) \approx (0, \vec{q})$, then $q^2 \approx -|\vec{q}|^2$

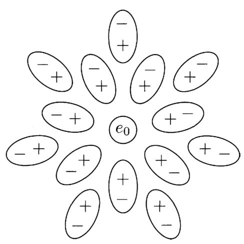
In this case $\Pi(-|\vec{q}|^2)$ is positive and increase with $|\vec{q}|^2$

$\Rightarrow$ The effective interaction $e_{eff}(q)$ increase with $|\vec{q}|^2$.

Note the Fourier transform is indeed the electrostatic potential

$$V(\vec{r}) = \int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{-e^2}{|\vec{q}|^2} = -\frac{\alpha_{em}}{r}$$

The effect can be interpreted as if the vacuum were a medium, which screens the charge, here through the creation of virtual e^+e^- pairs.



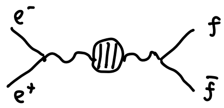
In this sense $\Pi(q^2)$ corresponds to vacuum polarization by the point-like electron

$\alpha = e^2/4\pi = 1/137.036\ldots$

corresponds to $q^2 = 0$, i.e. $r \to +\infty$

$e_{eff}(r) < e_{eff}(r=0)$

This effect is well measured in



and it is found that

$$\alpha_{eff}(0) = 1/137.036\ldots \to \alpha_{eff}(\underset{\substack{\uparrow \\ 91.8\,GeV}}{M_Z}) = 1/128.927$$

4.5 Renormalization of QCD

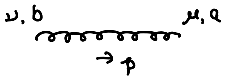
The SU(N) invariant Lagrangian for a set of fermions with non-Abelian gauge fields

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + (\partial^\mu \bar{c}^a)(\delta^{ac}\partial_\mu + g f^{abc} A^b_\mu) c^c$$
$$+ \bar{\psi}_i(\delta_{ij}\, i\not{\partial} + g \not{A}^a T^a_{ij} - m\delta_{ij})\psi_j$$

where c^a and $\bar{c}^a$ are the Faddeev – Popov ghosts and anti-ghost

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \qquad [\, a = 1, \cdots, N^2-1 \,]$$

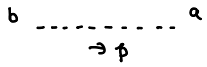
The gluon propagator:



$$= \frac{i\delta^{ab}}{p^2+i\varepsilon}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]$$

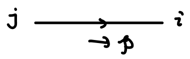
sum over all possible gluons, which gives a sum over a and b

The ghost propagator:



$$= \frac{i\delta^{ab}}{p^2+i\varepsilon}$$

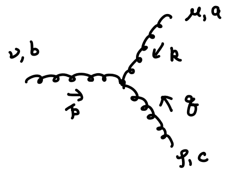
The propagators for coloured fermions:



$$= \frac{i\delta^{ij}}{\not{p}-m+i\varepsilon} \qquad [\, i, j \text{ refer to fundamental color indices} \,]$$

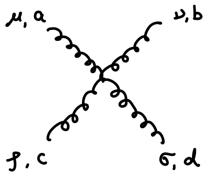
δ_{ij} or δ_{ab} indicates "color is conserved"

The triple - gluon vertex: [all momentum incoming : $p+k+q=0$]



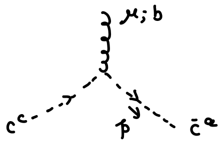
$$= g f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right]$$

The four-gluon vertex:



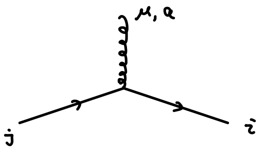
$$= -ig^2 \Big[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \Big]$$

The ghost vertex



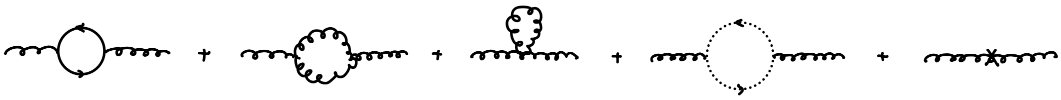
$$= -g f^{abc} p^\mu$$

The fermion interaction:



$$= i g \gamma^\mu T^a_{ij}$$

$\boxed{\text{Vacuum polarization:}}$

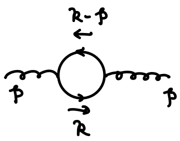


Fermion loop   gauge boson loop   ghost loop   counterterm

$$\mathcal{M}^{ab,\mu\nu} = \mathcal{M}_F + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_{gh} + \mathcal{M}_{c.t.}$$

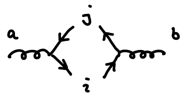
<u>Fermion bubble:</u>

$$i\mathcal{M}_F^{ab,\mu\nu} =$$



$$= -\,\mathrm{tr}[T^a T^b]\,(ig)^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{i}{(p-k)^2 - m^2 + i\varepsilon}\, \frac{i}{k^2 - m^2}\, \mathrm{Tr}\left[\gamma^\mu(\not{k} - \not{p} + m)\gamma^\nu(\not{k} + m)\right]$$

$T^a_{ij}\ T^b_{ji}$:



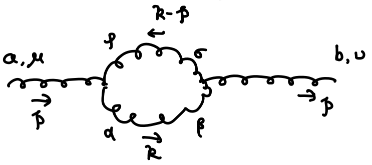
the color factor $\mathrm{tr}[T^a T^b] = T_F\,\delta^{ab}$ with $T_F = 1/2$

Then

$$\mathcal{M}_F^{ab,\mu\nu} = -\,\delta^{ab}\, T_F\, \frac{g^2}{2\pi^2}\,(p^2 g^{\mu\nu} - p^\mu p^\nu) \times \int_0^1 dx\ x\,\tilde{x} \left[\frac{2}{\varepsilon} + \ln\frac{\tilde{\mu}^2}{m^2 - p^2 x(1-x)}\right]$$

$$\underset{m\to 0}{=}\ \delta^{ab}\, T_F \left(\frac{g^2}{16\pi^2}\right)(p^2 g^{\mu\nu} - p^\mu p^\nu)\left[-\frac{4}{3\varepsilon} - \frac{20}{9} - \frac{4}{3}\ln\frac{\mu^2}{-p^2}\right]$$

Gluon bubble [$\xi = 1$, Feynman gauge]

$i\mathcal{M}_3^{ab,\mu\nu} =$ 

$$= \frac{1}{2} g^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2} \frac{-i}{(k-p)^2} \underbrace{f^{ace} f^{bdf} \delta^{cf} \delta^{ed}}_{f^{ace} f^{bec} = -C_A \delta^{ab}} N^{\mu\nu}$$

symmetry factor

The numerator $N^{\mu\nu} = [\, g^{\mu\alpha}(p+k)^\rho + g^{\alpha\rho}(p-2k)^\mu + g^{\rho\mu}(k-2p)^\alpha \,]\, g^{\alpha\beta} g^{\rho\sigma}$

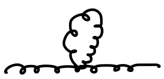
$$\times [\, g^{\nu\rho}(p+k)^\sigma - g^{\beta\sigma}(2k-p)^\nu - g^{\sigma\nu}(2p-k)^\beta \,]$$

Then

$$\mathcal{M}_3^{ab,\mu\nu} = -\frac{g^2}{2} \frac{\tilde{\mu}^{4-d}}{(4\pi)^{d/2}} \delta^{ab} C_A \int_0^1 dx \left(\frac{-1}{x\bar{x}p^2} \right)^{2-\frac{d}{2}}$$

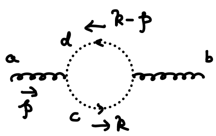
$$\times \Big\{ -g^{\mu\nu}\, 3(d-1)\, \Gamma(1-\tfrac{d}{2})\, x\bar{x}p^2 + p^\mu p^\nu \left[6(x^2-x+1) - d(1-2x)^2 \right] \Gamma(2-\tfrac{d}{2})$$

$$+ g^{\mu\nu} p^2 \left[(-2x^2+2x-5)\, \Gamma(2-\tfrac{d}{2}) \right] \Big\}$$

$i\mathcal{M}_4^{ab,\mu\nu} =$  $\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} = 0$

scaleless integral in dim-reg.

Ghost bubble:

$i\mathcal{M}_{gh}^{ab,\mu\nu} =$ 

$= (-1)(-g^2) \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k-p)^2} \frac{i}{k^2} f^{cad} k^\mu f^{dbc} (k-p)^\nu$

anticommutes

$$\Rightarrow \quad \mathcal{M}_{gh}^{ab,\mu\nu} = g^2 \frac{\tilde{\mu}^{4-d}}{(4\pi)^{d/2}} \delta^{ab} C_A \int_0^1 dx \left(\frac{1}{-x\bar{x}p^2} \right)^{2-\frac{d}{2}}$$

$$\times \Big\{ -g^{\mu\nu} \left[\tfrac{1}{2} \Gamma(1-\tfrac{d}{2})\, x\bar{x}p^2 \right] + p^\mu p^\nu \left[x\bar{x}\, \Gamma(2-\tfrac{d}{2}) \right] \Big\}$$

Adding all the gluon and ghost graphs

$$\mathcal{M}^{\mu\nu,ab}_{glue} = \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_{gh}$$

$$= C_A \delta^{ab} \frac{g^2}{16\pi^2} \underbrace{(g^{\mu\nu} p^2 - p^\mu p^\nu)}_{\text{satisfy the Ward identity}} \left[\frac{5}{3\varepsilon} + \frac{31}{9} + \frac{5}{3} \ln \frac{\mu^2}{-p^2} + O(\varepsilon) \right]$$

Adding the fermion graphs with n_f flavors.

$$\mathcal{M}^{ab,\mu\nu} = \delta^{ab} \frac{g^2}{16\pi^2} (g^{\mu\nu} p^2 - p^\mu p^\nu) \left[C_A \left(\frac{5}{3\varepsilon} + \frac{5}{3} \ln \frac{\mu^2}{-p^2} \right) - n_f T_F \left(\frac{4}{3\varepsilon} + \frac{4}{3} \ln \frac{\mu^2}{-p^2} \right) \right]$$

Renormalization at one loop

$$\mathcal{L} = \mathcal{L}_r + \mathcal{L}_{ct}$$

$$= -\frac{1}{4} Z_3 (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + Z_2 \bar{\psi}_i (i\not\partial - Z_m m) \psi_i - Z_{3c} \bar{c}^a \partial^2 c^a$$

$$- g Z_{A^3} f^{abc} (\partial_\mu A^a_\nu) A^{b,\mu} A^{c,\nu} - \frac{1}{4} g^2 Z_{A^4} (f^{eab} A^a_\mu A^b_\nu)(f^{ecd} A^{c,\mu} A^{d,\nu})$$

$$+ g Z_1 A^a_\mu \bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j + g Z_{1c} f^{abc} (\partial_\mu \bar{c}^a) A^{b,\mu} c^c$$

We will work in $\overline{MS}$ scheme.

The UV - pole of a integral

$$\left[\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} \right]_{UV-div} = \frac{i}{8\pi^2} \frac{1}{\varepsilon}$$

<u>2-pt fns:</u>

gluon vacuum polarization:

$$\mathcal{M}^{ab,\mu\nu} = \delta^{ab} (g^{\mu\nu} p^2 - p^\mu p^\nu) \left\{ \frac{g^2}{16\pi^2} \left[C_A \frac{5}{3\varepsilon} - n_f T_F \frac{4}{3\varepsilon} \right] - (Z_3 - 1) \right\}$$

 $-i (p^2 g^{\mu\nu} - p^\mu p^\nu) \delta^{ab} (Z_3 - 1)$

$$\Rightarrow \quad Z_3 - 1 = \frac{1}{\varepsilon} \frac{g^2}{16\pi^2} \left[\frac{5}{3} C_A - \frac{4}{3} n_f T_F \right]$$

For Z_2 and Z_m, one needs the quark self energy graphy:

$i \Sigma_2^{ij}(\not p) =$  $+$ 

$$= \sum_a (T^a T^a)_{ji} (ig)^2 \int \frac{d^4 k}{(2\pi)^4} \gamma^\mu \frac{i(\not k + m)}{k^2 - m^2 + i\varepsilon} \gamma_\mu \frac{-i}{(k-p)^2 + i\varepsilon} + i\mathcal{M}\, c.t.$$

color factor: $= C_F \delta_{ij} = \frac{N^2 - 1}{2N}$

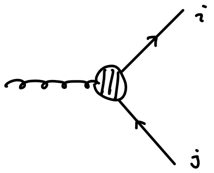
$$\Rightarrow \quad \Sigma_2^{ij}(\not p) = \delta^{ij} \left\{ \frac{g^2}{16\pi^2} C_F \frac{\not p - 4m}{\varepsilon} + \text{finite} + (Z_2 - 1)\not p - (Z_m Z_2 - 1) m \right\}$$

$$\Rightarrow \quad Z_2 - 1 = -\frac{C_F}{\varepsilon} \frac{g^2}{16\pi^2} \qquad Z_m - 1 = -\frac{3 C_F}{\varepsilon} \frac{g^2}{16\pi^2}$$

For Z_{3c} one needs the ghost 2-pt function

$$Z_{3c} - 1 = \frac{C_A}{2\varepsilon}\frac{g^2}{16\pi^2}$$

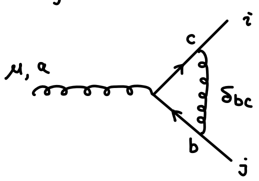
<u>3-pt fns:</u>



$$\equiv ig\,\Gamma^{a,\mu}_{ij}$$

At one loop, there are two contributions:

(I)



$$= ig\,(T^c T^a T^b)_{ij}\,\delta^{bc} \times \Gamma^{\mu}_{(2A)}$$

color factor: $T^b T^a T^b$

$$\begin{aligned}
&= T^b T^b T^a + T^b [T^a, T^b] \\
&= C_F T^a + i f^{abc} T^b T^c \\
&= C_F T^a + \frac{i}{2} f^{abc} [T^b, T^c] \\
&= C_F T^a - \frac{1}{2} f^{abc} f^{bcd} T^d \\
&= (C_F - \frac{C_A}{2})\, T^a
\end{aligned}$$

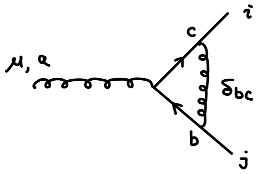
$\Gamma^{\mu}_{(2A)}$ is the same as in QED:

$$\Gamma^{\mu}_{(2A)} = F_1^{(2A)}(q^2, m^2)\,\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m} q^\nu \underbrace{F_2^{(2A)}(q^2, m^2)}_{\text{finite}}$$

Then the Dirac form factor:

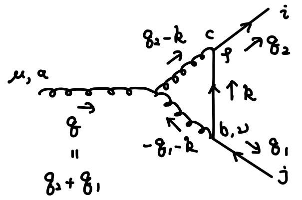
$$F_1^{(2A)} = \frac{g^2}{8\pi^2}\left\{ \frac{1}{2\varepsilon} - \frac{1}{2} + \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\left[\frac{p^2(1-x)(1-y) + m^2(1-4z+z^2)}{\Delta} + \ln\frac{\mu^2}{\Delta}\right]\right\}$$

with $\Delta = (1-z)^2 m^2 - xy\,p^2$



$$\xlongequal{p^2 \gg m^2} ig\,(C_F - \frac{C_A}{2})\, T^a_{ij}\,\gamma^\mu \frac{g^2}{16\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{-q^2} + \text{finite}\right)$$

(II)



$$= ig\,\underbrace{f^{abc}\,(T^cT^b)_{ij}}\,\Gamma^\mu_{(2B)}$$

color factor : $T^cT^bf^{abc}$

$$= \frac{1}{2}f^{abc}[T^c, T^b]$$

$$= -\frac{i}{2}f^{abc}f^{dbc}T^d$$

$$= -\frac{i}{2}C_A\,T^a$$

$$(ig)\,\Gamma^\mu_{(2B)}(p^2, m^2=0) = (ig)^2 g\int\frac{d^4k}{(2\pi)^4}\gamma^\rho\frac{i\not{k}}{k^2}\gamma^\nu\frac{-i}{(q_1+k)^2+i\varepsilon}\,\frac{-i}{(q_2-k)^2+i\varepsilon}$$

$$\times[g^{\mu\nu}(2q_1+q_2+k)^\rho + g^{\nu\rho}(-q_1+q_2-2k)^\mu + g^{\rho\mu}(k-2q_2-q_1)^\nu]$$

to extract the UV divergences, set all the external momenta to zero :

$$\hookrightarrow\ \Gamma^\mu_{(2B)} \approx g^2\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\rho\not{k}\gamma^\nu}{k^6}[g^{\mu\nu}k^\rho - 2g^{\nu\rho}k^\mu + g^{\rho\mu}k^\nu]$$

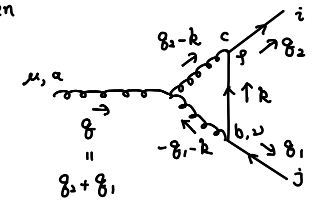
$$= g^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^6}[2k^2\gamma^\mu - 2\gamma^\rho\not{k}\gamma^\rho k^\mu]$$

in dim.-reg. $k^\mu k^\nu \to \frac{k^2}{d}g^{\mu\nu}$, $\gamma^\rho\gamma^\nu\gamma^\rho = (2-d)\gamma^\nu$

$$= \left(4-\frac{4}{d}\right)\gamma^\mu g^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^4}$$

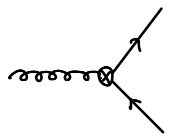
$$= i\gamma^\mu\frac{g^2}{8\pi^2}\left(\frac{3}{2\varepsilon}+\frac{3}{2}\ln\mu^2+\dots\right)$$

Then



$$= ig\,C_A T^a_{ij}\gamma^\mu\left(\frac{g^2}{16\pi^2}\right)\left(\frac{3}{2\varepsilon}+\frac{3}{2}\ln\frac{\mu^2}{-p^2}+\text{finite}\right)$$

Finally, the counterterm :



$$= ig\,T^a_{ij}\gamma^\mu(Z_1-1)$$

Then

$$(Z_1-1) = \frac{1}{\varepsilon}\frac{g^2}{16\pi^2}(-C_F-C_A)$$

Summarize all the counterterms in QCD at one loop. [arbitrary R_ξ gauge]

$$\delta_1 \equiv Z_1 - 1 = \frac{1}{2\varepsilon}\frac{g^2}{16\pi^2}\left[-2C_F - 2C_A + 2(1-\xi)C_F + \frac{1}{2}(1-\xi)C_A \right]$$

$$\delta_2 \equiv Z_2 - 1 = \frac{1}{2\varepsilon}\frac{g^2}{16\pi^2}\left[-2C_F + 2(1-\xi)C_F \right]$$

$$\delta_m \equiv Z_m - 1 = \frac{1}{2\varepsilon}\frac{g^2}{16\pi^2}\left[-6C_F \right]$$

$$\delta_3 \equiv Z_3 - 1 = \frac{1}{2\varepsilon}\frac{g^2}{16\pi^2}\left[\frac{10}{3}C_A - \frac{8}{3}T_F n_f + (1-\xi)C_A \right]$$

$$\delta_{3c} \equiv Z_{3c} - 1 = \frac{1}{2\varepsilon}\frac{g^2}{16\pi^2}\left[C_A + \frac{1}{2}(1-\xi)C_A \right]$$

$$\delta_{A^3} = Z_{A^3} - 1 = \frac{1}{2\varepsilon}\frac{g^2}{16\pi^2}\left[\frac{4}{3}C_A - \frac{8}{3}n_f T_F + \frac{3}{2}(1-\xi)C_A \right]$$

$$\delta_{A^4} = Z_{A^4} - 1 = \frac{1}{2\varepsilon}\frac{g^2}{16\pi^2}\left[-\frac{2}{3}C_A - \frac{8}{3}n_f T_F + 2(1-\xi)C_A \right]$$

$$\delta_{1c} = Z_{1c} - 1 = \frac{1}{2\varepsilon}\frac{g^2}{16\pi^2}\left[-C_A + (1-\xi)C_A \right]$$

<u>QCD β-function:</u>

$$\mathcal{L}_{int} = g_0 \bar\psi_0 \not{A}_0^a T^a \psi_0 \qquad \psi_0 = \sqrt{Z_2}\,\psi \ , \quad A_0^a = \sqrt{Z_3}\,A^a$$

$$= g\tilde\mu^\varepsilon Z_1 \bar\psi \not{A}^a T^a \psi$$

$$= g\tilde\mu^\varepsilon \frac{Z_1}{Z_2\sqrt{Z_3}} \bar\psi_0 \not{A}_0^a T^a \psi_0$$

$\Rightarrow$ the bare coupling: $g_0 = g\tilde\mu^\varepsilon \dfrac{Z_1}{Z_2\sqrt{Z_3}}$ which is independent of μ

so $\quad 0 = \mu\dfrac{d}{d\mu} g_0 = \mu \dfrac{d}{d\mu}\left[g\tilde\mu^\varepsilon \dfrac{Z_1}{Z_2\sqrt{Z_3}} \right]$

$$\hookrightarrow \beta(g) = \mu\frac{d}{d\mu} g = g\left[-\varepsilon - \mu\frac{d}{d\mu}\left(\delta_1 - \delta_2 - \frac{\delta_3}{2}\right) \right]$$

each δ depends on μ through g , $\quad \mu\dfrac{d}{d\mu} = \mu\dfrac{\partial}{\partial g}\dfrac{dg}{d\mu} = \beta(g)\dfrac{\partial}{\partial g} = -\varepsilon g \dfrac{\partial}{\partial g} + O(g^3)$

$$\hookrightarrow \quad \beta(g) = -\varepsilon g + \varepsilon g^2 \frac{\partial}{\partial g}(\delta_1 - \delta_2 - \delta_3/2)$$

$$= -\varepsilon g + g^2 \frac{g}{16\pi^2}\Big[-2C_F - 2C_A + 2(1-\xi)C_F + \frac{1-\xi}{2}C_A$$

$$+2C_F - 2(1-\xi)C_F$$

$$-\frac{5}{3}C_A + \frac{4}{3}n_f T_F - \frac{1}{2}(1-\xi)C_A \Big]$$

$$= -\varepsilon g - \frac{g^3}{16\pi^2}\Big[\frac{11}{3}C_A - \frac{4}{3}n_f T_F\Big] \qquad \text{"}\xi \text{ dependence cancells"}$$

Similarly, one can compute the β function in th A^3 interaction:

$$\beta(g) = -\varepsilon g + \varepsilon g^2 \frac{\partial}{\partial g}\left(\delta_{A^3} - \frac{3}{2}\delta_3\right)$$

which is the same as before due to gauge invariance. As shown in the one loop calculation of the vaccuum polarization, the relative size of the gluon 3-pt and 4-pt self interactions is crucial to the Ward identity. The only way all of the factors of g will be renormalized in the same way is

$$\frac{Z_1}{Z_2} = \frac{Z_{1c}}{Z_{3c}} = \frac{Z_{A^3}}{Z_3} = \frac{\sqrt{Z_{A^4}}}{\sqrt{Z_3}}$$

At one loop

$$\delta_1 - \delta_2 = \delta_{1c} - \delta_{3c} = \delta_{A^3} - \delta_3 = \frac{1}{2}(\delta_{A^4} - \delta_3) = -\frac{1}{2\varepsilon}\frac{g^2}{32\pi^2} C_A(\xi+3)$$

In QCD, $N=3$, $C_A = 3$, let $\alpha_s = \frac{g^2}{4\pi}$ then

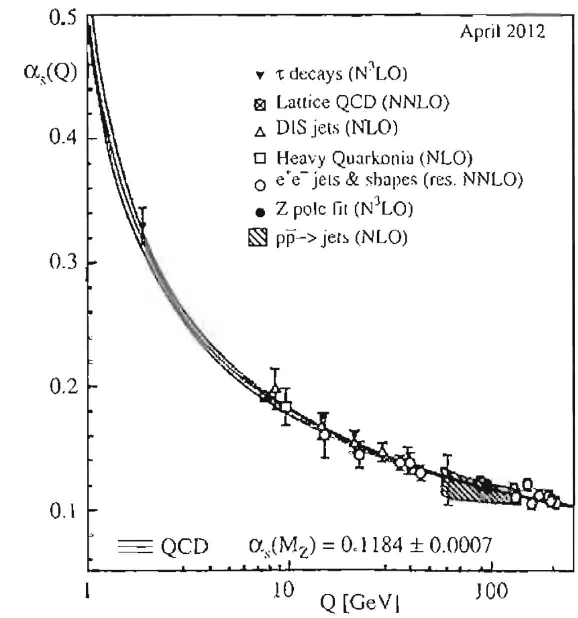
$$\mu \frac{d}{d\mu}\alpha_s(\mu) = -\frac{\alpha_s^2}{2\pi}\beta_0 \qquad \text{with } \beta_0 = 11 - \frac{2}{3}n_f$$

$$\beta_0 > 0 \quad \text{as } n_f < 17$$

Then the solution is

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0}\frac{1}{\ln\frac{\mu}{\Lambda_{QCD}}} \qquad \alpha_s(\Lambda_{QCD}) \to \infty \quad , \quad \Lambda_{QCD} \sim 200\ \text{MeV}$$

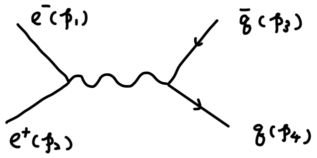
Asymptotic freedom: α_s gets weaker at high energy.


April 2012
$\alpha_s(Q)$
τ decays (N^3LO)
Lattice QCD (NNLO)
DIS jets (NLO)
Heavy Quarkonia (NLO)
e^+e^- jets & shapes (res. NNLO)
Z pole fit (N^3LO)
$p\bar{p}$–> jets (NLO)
QCD
$\alpha_s(M_Z) = 0.1184 \pm 0.0007$
0.5
0.4
0.3
0.2
0.1
1
10
100
Q [GeV]

$\boxed{e^+e^- \to \text{hadrons}}$

The simplest quantity that can be evaluated perturbatively in QCD is the total cross section $e^+e^- \to$ hadrons.

<u>$e^+e^- \to q\bar{q}$</u>



$$\sigma^{tot} = \frac{\pi\alpha^2}{2s}\,\frac{8}{3}\,N_c\sum_q e_q^2$$

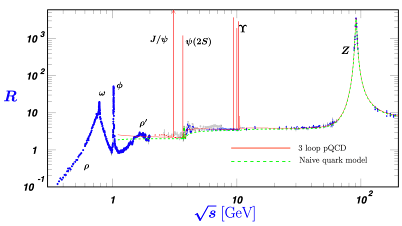
It is common to look at the ratio

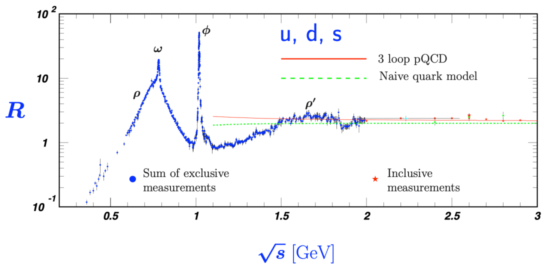
$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

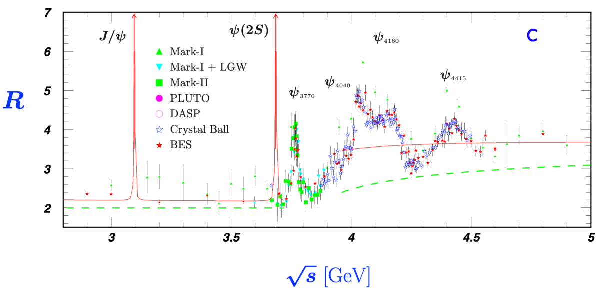
$$= N_c \sum_q e_q^2 \left\{ 1 + O(\alpha_s) \right\} \qquad [\text{neglect the fermion mass}]$$

Looking at the data, the agreement is quite nice, except at very low energy or near thresholds $\sqrt{s} \approx 2m_Q$, where Q is charm or bottom quark.

e^+e^- to hadrons: R-ratio (PDG)





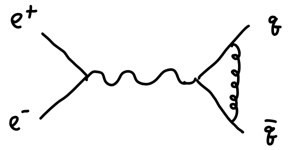


The R-ratio provides a great illustration of asymptotic freedom: the hadronic cross section at high energies is well approximated by the cross section e^+e^- to a pair of free quarks.

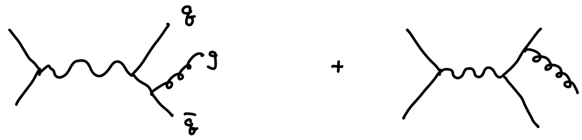
Let us now look at perturbative corrections

At $O(\alpha_s)$, there are two types of corrections:

1) Virtual corrections: one-loop corrections to $e^+e^- \to q\bar{q}$



2) Real emissions



One finds that

$$R = N_c \sum_q e_q^2 \left\{ 1 + \frac{\alpha_s}{\pi} \right\}$$

At the next order, evaluating the diagrams gives

$$R = N_c \sum_q e_q^2 \left\{ 1 + \frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2 \cdot \left[C_2 + \frac{\beta_0}{4}\left(\frac{1}{\varepsilon} - \ln s\right) \right] \right\}$$

with $C_2 = \left(\frac{2}{3}\zeta_3 - \frac{11}{12}\right) n_f + \frac{365}{24} - 11\zeta_3$

$$= 1.99 - 0.115\, n_f$$

Here $\alpha_s \equiv \alpha_s^0$ the bare coupling:

Renormalize $\alpha_s^0 = \alpha_s(\mu)\, Z_g^2\, \tilde{\mu}^{2\varepsilon}$

$$= \alpha_s(\mu) \left[1 - \frac{\alpha_s}{4\pi}\frac{\beta_0}{2\varepsilon} \right]^2 \tilde{\mu}^{2\varepsilon}$$

$$\hookrightarrow \quad R = N_c \sum_q e_q^2 \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left[C_2 - \frac{\beta_0}{4} \ln\frac{s}{\mu^2} \right] \right\}$$

Note:

$$\frac{d}{d\ln\mu}\left[\frac{\alpha_s(\mu)}{\pi}\right] = \frac{1}{\pi}\left[-2\,\alpha_s(\mu)\,\beta_0\,\frac{\alpha_s}{4\pi}\right] = \left(\frac{\alpha_s}{\pi}\right)^2\left[-\frac{\beta_0}{2}\right]$$

$$\hookrightarrow \quad \frac{d}{d\ln\mu}R = N_c \sum_q e_q^2 \left\{ \left(\frac{\alpha_s}{\pi}\right)^2\left(-\frac{\beta_0}{2}\right) + \frac{\beta_0}{2}\left(\frac{\alpha_s}{\pi}\right)^2 \right\} + o(\alpha_s^3)$$

$$= o(\alpha_s^3)$$

So R is scale invariant up to the order we have calculated.

However, if we choose $\mu^2 \gg s$ or $\mu^2 \ll s$, the $\ln\frac{s}{\mu^2}$ term will become very large and perturbation theory break down, so we should choose $\mu^2 \approx s$.

Operator product expansion in e^+e^-

The operator product expansion (OPE) is a very useful tool with many application in QFT. In our case, it will allow us to show

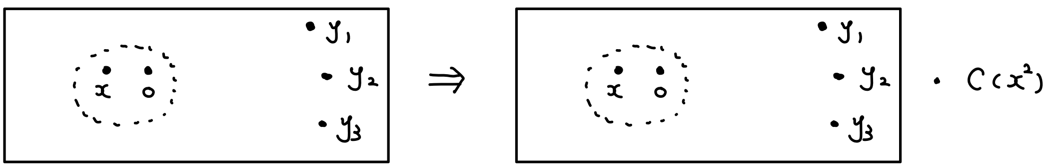
1°. Our calculation of the cross section in terms of quarks and gluons can be justified.

2°. What the nonperturbative corrections to our result are

General idea: Consider product of two operators $O_1(x)\, O_2(0)$. Now look at Green functions.

$$G(x, 0, y_1 \cdots y_n) = \langle O_1(x)\, O_2(0)\, \phi(y_1) \cdots \phi(y_n) \rangle$$

In the limit of $y_i^2 \gg x^2$ (in Euclidean space) we should be able to replace $O_1(x)\, O_2(0)$ by a local operator times of a function of x^2



Since this should hold for any Green's function, the relation should be an operator relation.

Wilson proposed '69

$$\overset{ren}{O}(x)\, \overset{ren}{O}(0) \underset{x\to 0}{=} \sum_n C_n(x^2, \mu^2)\, O_n^{ren}(0)$$

There are singularities in the limit $x \to 0$

absorb into C_n

Note: The operators O_n have the same quantum numbers as the product and it is useful to order the operators by their dimension.

For example:

$$\phi(x)\,\phi(0) = C_1\,\mathbb{1} + C_{\phi^2}\,\phi^2(0) + C_{\phi^4}\,\phi^4(0) + C_{(\partial\phi)^2}\,\partial_\mu\phi\,\partial^\mu\phi + \cdots$$

Since $\phi(x)$ has dimension of mass:

$$C_1 \sim 1/x^2, \quad C_{\phi^2} \sim 1, \quad C_{\phi^4} \sim x^2$$

The lowest dimensional operators have the most singular coefficients as $x \to 0$

The higher dimension operators are suppressed.

We can thus approximate the operator product by the first few operators on the RHS.

The naive dimensional analysis is slightly modified by renormalization, since $C_i = C_i(x^2, \mu)$ but as in the case of the coupling constant, the μ-dependence can be controlled using a RG equation. In perturbation theory higher order corrections will involve $\ln(x^2\mu^2)$ terms.

The optical theorem.

The optical theorem allows us to rewrite the total cross section as the imaginery part of the forward scattering amplitude. In this form, we will then apply the OPE.

It follows from the unitary of the S-matrix , $S^{\dagger} S = 1$.

Write $\quad S = 1 + iT$.

$$\hookrightarrow \quad (1 - iT^{\dagger})(1 + iT) = 1$$

$$\hookrightarrow \quad \underbrace{-i(T - T^{\dagger})}_{2\,\mathrm{Im}} = T^{\dagger} T$$

Rewrite the RHS:

$$\langle p_1 p_2 | T^{\dagger} T | k_1 k_2 \rangle = \sum_{X} \langle p_1 p_2 | T^{\dagger} | X \rangle \langle X | T | k_1 k_2 \rangle$$

$$\hookrightarrow \quad 2\,\mathrm{Im}\,\mathcal{M}(k_1 k_2 \to p_1 p_2) = \sum_{X} \mathcal{M}^{*}(p_1 p_2 \to p_X)\, \mathcal{M}(k_1 k_2 \to p_X)\, (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_X)$$

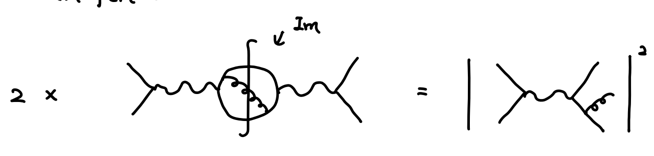
To get the standard form set $k_1 = p_1$, $k_2 = p_2$

$$\hookrightarrow \quad 2\,\mathrm{Im}\,\mathcal{M}(p_1 p_2 \to p_1 p_2) = \sum_{X} |\mathcal{M}(p_1 p_2 \to p_X)|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_X)$$

$$= 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}\;\sigma^{tot}$$

$$\hookrightarrow \quad \mathrm{Im}\,\mathcal{M} = 2 E_{cm}\, p_{cm}\, \sigma^{tot}$$

Pictorically :

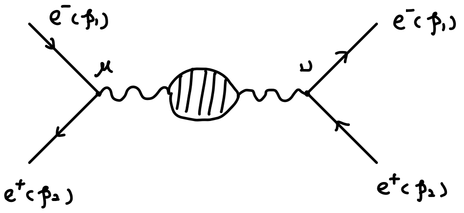


For $e^+ e^-$, neglecting m_e, we have

$$\sigma_{e^+e^-} = \frac{1}{s}\,\mathrm{Im}\,\mathcal{M}(e^+e^- \to e^+e^-)$$

Operator analysis of e^+e^-

The diagrams for $e^+e^- \to e^+e^-$ whose imaginary part gives $e^+e^- \to$ hadrons



The amplitude is

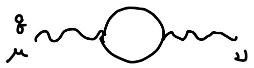
$$i\mathcal{M} = (-ie)^2 \; \bar{v}(p_2)\gamma^\mu u(p_1) \; \bar{u}(p_1)\gamma^\nu v(p_2) \; \frac{-i}{s} \; i\Pi^h_{\mu\nu}(q) \; \frac{-i}{s}$$

$\Pi^h_{\mu\nu}$ is the hadronic part of the self energy and has the form

$$\Pi_h^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\,\Pi_h(q^2)$$

$\hookrightarrow \quad \sigma(e^+e^- \to \text{hadrons}) = -\frac{4\pi\alpha}{s}\,\mathrm{Im}\,\Pi_h(s)$

As a check, we should now evaluate the photon self energy:



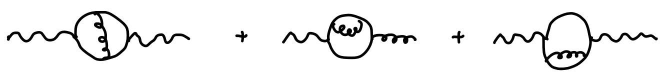
one finds $\quad \Pi(s) = -\frac{\alpha}{3} N_c \sum_q e_q^2 \left[-\frac{1}{\pi} \ln(-s-i\varepsilon)\right]$

$$\mathrm{Im}\,\Pi(s+i\varepsilon) = -\frac{\alpha}{3} N_c \sum_q e_q^2$$

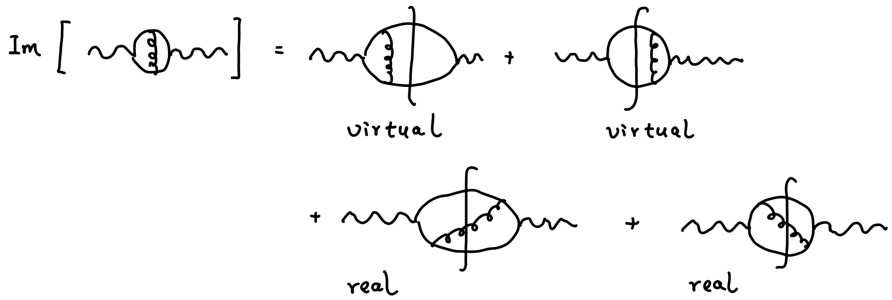
$\Rightarrow \quad \sigma(e^+e^- \to q\bar{q}) = \frac{4\pi\alpha^2}{3s} N_c \sum_q e_q^2$

So we now get the cross section from a loop calculation <u>alone</u>.

At $O(\alpha_s)$



The IR div. cancellation between real and virtual is manifest:



"The imaginary part arises when particles in the loop go on the mass shell."

The electromagnetic coupling of quarks has the form

$$J^{\mu} = \sum_{q} e_q \, \bar{q}_f \gamma^{\mu} q_f$$

So the self energy:

$$i \Pi_h^{\mu\nu}(q) = (ie)^2 \int d^4x \, e^{iq\cdot x} \langle \Omega | T\{ J^{\mu}(x) J^{\nu}(0) \} | \Omega \rangle$$

In the limit $x \to 0$, we can expand the product of currents:

$$J_{\mu}(x) J_{\nu}(0) = C^{1}_{\mu\nu}(x) + C^{q\bar{q}}_{\mu\nu}(x) \, m_q \bar{q} q + C^{G^2}_{\mu\nu}(x) \, [G^{a}_{\mu\nu}]^2$$

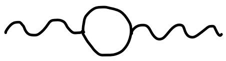
with $\quad C^{1}_{\mu\nu} \sim (x^2)^{-3}, \quad C^{q\bar{q}}_{\mu\nu} \sim (x^2)^{-1}, \quad C^{G^2}_{\mu\nu} \sim (x^2)^{-1}$

If the Fourier transform is dominated by $x^2 \approx 0$, we get

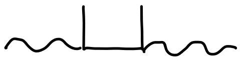
$$i \Pi_h^{\mu\nu}(q) = -ie^2 (q^2 g^{\mu\nu} - q^{\mu} q^{\nu}) [C^{1}(q^2) + C^{q\bar{q}}(q^2) \, m \bar{q} q + C^{G^2}(q^2) (G^{a}_{\mu\nu})^2]$$

$$C^{1} \sim (q^2)^0 \; , \quad \underbrace{C^{q\bar{q}} \sim 1/q^2 \; , \quad C^{G^2} \sim 1/q^2}_{\text{suppressed at high energy}}$$

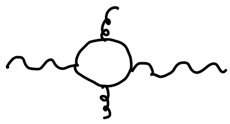
Since the OPE is an operator relation, we can take an arbitrary matrix element to obtain the Wilson coefficients. In particular, we can work with quark and gluon states.



$\longrightarrow \quad C^1(q^2)$



$\longrightarrow \quad C^{q\bar{q}}(q^2)$



$\longrightarrow \quad C^{G^2}(q^2)$

Then, with the coefficients determined from perturbative calculations, we take the physical vaccum matrix element.

↳
$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{s}\Big[\ \mathrm{Im}\, C^1(q^2) + \mathrm{Im}\, C^{q\bar{q}}(q^2) \underbrace{\langle\Omega|\, m\,\bar{q}q\,|\Omega\rangle}_{\sim (0.3\,\mathrm{MeV})^4 \ \text{quark condensate}} + \mathrm{Im}\, C^{G^2}(q^2)\ \underbrace{\langle\Omega|\,(G^a_{\mu\nu})^2\,|\Omega\rangle}_{\sim \text{Gluon condensate } (0.5\,\mathrm{GeV})^4}$$

We have seperated the perturbative part (Wilson coefficient) from the non-perturbative part (operator matrix element).

So, up to corrections of order Λ_{QCD}/q^2 from the condensates, the hadron cross section is equal to the partonic cross section from perturbation theory.