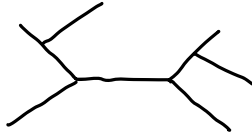


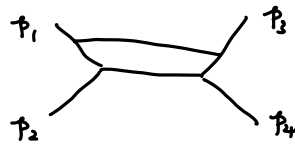
4. Renormalization

The perturbation expansion is visualized by Feynman diagrams.

"tree diagrams"



"loop diagrams"



involves $\int \frac{d^4k}{(2\pi)^4} I(p_1, p_2, p_3; k)$

- Loop diagrams are usually (not always) corrections to a process that already exists "at tree level"
- It turns out that the loop integrals $\int d^4k$ are in general divergent.

Renormalization theory deals with the correct interpretation of these seemingly ill-defined expression. The result is a much deeper understanding of QFT that reaches far beyond the realm of perturbation theory.

4.1 Regularization and renormalization theory

Develop the basics of renormalization theory for ϕ^4 theory.

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_0 \partial^\mu \phi_0 - m_0^2 \phi_0^2) - \frac{\lambda_0}{4!} \phi_0^4$$

The index "0" refers to the "bare" or "unrenormalized" parameters to distinguish them from the renormalized ones to be defined later.

A first example : the self energy

$$\int d^4x e^{i\vec{p} \cdot x} \langle \Omega | T \{ \phi_0(x) \phi_0(0) \} | \Omega \rangle = \rightarrow \text{diagram: a shaded circle with two external lines} \rightarrow$$

$$= \frac{1}{\vec{p}} + \frac{1}{\vec{p}} \text{diagram: circle with 1PI} \frac{1}{\vec{p}} + \frac{1}{\vec{p}} \text{diagram: circle with 1PI} \frac{1}{\vec{p}} \text{diagram: circle with 2PI} \frac{1}{\vec{p}} + \dots$$

$$= \frac{i}{\vec{p}^2 - m_0^2 + i\epsilon} + \frac{i}{\vec{p}^2 - m_0^2 + i\epsilon} \underbrace{\left[-i\pi(\vec{p}^2, m_0^2) \right]}_{\equiv \text{diagram: circle with 1PI}} \frac{i}{\vec{p}^2 - m_0^2 + i\epsilon} + \dots$$

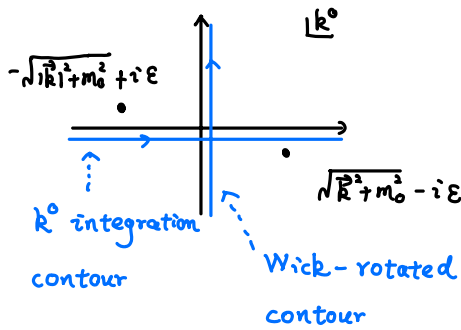
$$= \frac{i}{\vec{p}^2 - m_0^2 - \pi(\vec{p}^2, m_0^2) + i\epsilon}$$

In perturbation theory $\pi = \lambda \pi^{(1)} + \lambda^2 \pi^{(2)} + \dots$

$$\begin{aligned} -i\pi(\vec{p}^2, m_0^2) &= \text{diagram: circle with 1PI} \\ &= \text{diagram: tadpole with loop} + \text{diagram: bubble} + \text{diagram: sunset} + o(\lambda_0^3) \\ &\quad o(\lambda_0) \qquad \qquad o(\lambda_0^2) \end{aligned}$$

$$= \frac{1}{2} (-i\lambda_0) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} + o(\lambda_0^2)$$

Can do the k^0 -integral applying Cauchy's theorem or apply Wick rotation:



$$\begin{aligned}\int d^4k f(k^2) &= \int_{-\infty}^{+\infty} dk^0 \int d^3\vec{k} f(k^2) \\ &= \int_{-i\infty}^{+i\infty} dk^0 \int d^3\vec{k} f(k^2)\end{aligned}$$

Now define : $\vec{k}_E = \vec{k}$

$$k_E^0 = -ik^0$$

$$\text{then } = i \int_{-\infty}^{+\infty} dk_E^0 \int d^3\vec{k}_E f(-k_E^2)$$

$$= i \int d^4k_E f(-k_E^2)$$

$$[k_E^2 = (k_E^0)^2 + (\vec{k}_E)^2 = -k^2]$$

"Euclidean metric"

Hence

$$\begin{aligned}-i\pi(p^2, m_0^2) &= -i \frac{\lambda_0}{2} i \int \frac{d^4k_E}{(2\pi)^4} \frac{i}{-k_E^2 - m_0^2 + i\epsilon} \quad \text{can drop the } i\epsilon \text{ now,} \\ &\quad \text{since the contour is never} \\ &\quad \text{close to a pole.}\end{aligned}$$

$$= -i \frac{\lambda_0}{2} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{k_E^2 + m_0^2}$$

$$= \frac{-i\lambda_0}{32\pi^2} \int_0^{+\infty} dk_E^2 \frac{k_E^2}{k_E^2 + m_0^2}$$

$$= \infty$$

↑

"quadratically divergent"

since for large k_E , the integrand is $dk_E^2 \times 1$

$$\begin{aligned}\leftarrow \dots \int d^4k_E &= \int_0^{+\infty} dk_E^2 \frac{k_E^2}{2} \int d\Omega^{(4)} \quad \leftarrow \text{the area of a} \\ &= \pi^2 \int_0^{+\infty} dk_E^2 k_E^2 \quad \text{unit sphere in} \\ &\quad 4d.\end{aligned}$$

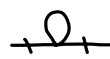
We define the integral by applying a regularization. For the moment, we choose a momentum cut-off: $|k_E| < \Lambda$

$$\Rightarrow -i\pi = \frac{-i\lambda_0}{32\pi^2} \left(\Lambda^2 - m_0^2 \log \frac{\Lambda^2 + m_0^2}{m_0^2} \right)$$

$$= \frac{-i\lambda_0}{32\pi^2} \left[\Lambda^2 - m_0^2 \log \frac{\Lambda^2}{m_0^2} + O\left(\frac{m_0^4}{\Lambda^2}\right) \right] \quad \text{for } \Lambda \gg m_0$$

$$\Rightarrow m^2 = m_0^2 + \frac{\lambda_0}{32\pi^2} (\Lambda^2 - m_0^2 \log \frac{\Lambda^2}{m_0^2} + \dots) + o(\lambda_0^2)$$

$$Z = 1 + o(\lambda_0^2) \quad [\text{This is special to } \phi^4 \text{ theory, because}$$

 does not depend on the external

$\Downarrow$

momentum p .]

The self-energy of the scalar field

and hence the physical mass of the

scalar particle is infinite, when $\Lambda \rightarrow \infty$!

The physical mass of the particle is given by the solution to

$$p^2 - m_0^2 - \pi(p^2, m_0^2) \Big|_{p^2=m^2} \stackrel{!}{=} 0$$

If π can be computed in perturbation theory, one can solve this equation iteratively to obtain

$$m^2 = m_0^2 + \delta m^2 \quad ; \quad \delta m^2 = \lambda \delta m^{2(1)} + o(\lambda^2)$$

$\uparrow$

mass renormalization $\leftrightarrow$ contribution to the mass from interaction

The field renormalization constant follows from expanding in p^2 around the pole

location m^2 :

$$\begin{aligned} p^2 - m_0^2 - \pi(p^2, m_0^2) &= \underbrace{m^2 - m_0^2 - \pi(p^2=m^2, m_0^2)}_{=0 \text{ by def of } m^2} + (p^2 - m^2) - (p^2 - m^2) \frac{\partial \pi}{\partial p^2} \Big|_{p^2=m^2} \\ &\quad + o((p^2 - m^2)^2) \end{aligned}$$

$$\Rightarrow \frac{i}{p^2 - m_0^2 - \pi(p^2, m_0^2)} \xrightarrow{p^2 \rightarrow m^2} \frac{i}{p^2 - m^2} \frac{1}{1 - \frac{\partial \pi}{\partial p^2} \Big|_{p^2=m^2}} + \text{non-singular}$$

$$\text{Hence } Z^{-1} = 1 - \frac{\partial \pi(p^2, m_0^2)}{\partial p^2} \Big|_{p^2=m^2}$$

The renormalization paradigm

- The divergence arise from loop momenta $k \gg$ ^{mass,}
external momenta
(also called ultra-violet divergence)

The integration over $\vec{k}$, k^0 up to ∞ arises in the first place, because the interactions are local (large $k \leftrightarrow$ small x) and the particles point-like (one d.o.f. at every $\vec{x}$)

- Empirically know

- 1) Fundamental interactions are local and relativistically invariant up to some scattering energy $E_{\max}$, or some $x_{\min} \sim 1/E_{\max}$. Can never probe arbitrarily large $E_{\max}$.

But $\mathcal{L}(\phi_0, m_0, \lambda_0)$ describes a theory, which is local and Lorentz-invariant for all energies, no matter how large, so it contains a large extrapolation.

- 2) Relations between observables in experiments at energies $\lesssim E_{\max}$, can not depend in an essential way on physics at much higher energies, otherwise one could resolve arbitrarily small structures with finite energy in contradiction to the uncertainty principle. Hence if $\mathcal{L}(\phi_0, m_0, \lambda_0)$ describes a sensible theory, the relations between physical quantities must be independent of the regularization.

But m_0, λ_0 are not by themselves physical quantities / directly measurable.

Only m is, as are scattering cross sections, etc.

- Hence conjecture that the following procedure can be applied

$$\mathcal{L}(\phi_0; m_0, \lambda_0)$$

$\downarrow$ replace m_0, λ_0 by physical quantities m, λ
 $m(m_0, \lambda_0), \lambda(m_0, \lambda_0)$ depend on the regularization

Compute other physical observables f from $\mathcal{L}$,

but express them in terms of m, λ , not m_0, λ_0 :

$$f = f(m, \lambda) = f(m_0(m, \lambda), \lambda_0(m, \lambda))$$

This should be independent of the regularization and represent an unambiguous, testable prediction of the theory.

λ_0, m_0 are therefore only auxiliary parameters. If the above interpretation is correct, any regularization method can be chosen and one chooses the one that makes computations simple. Once the regularization is removed, the theory may or may not describe phenomena up to arbitrarily high energy. It does not matter. We will never know (= be able to test this by observation) anyway.

Regularization methods and Feynman parameters

Consider the integral:

$$A(a; \Delta) \equiv \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\varepsilon)^a} = (-1)^a i \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{(k_E^2 + \Delta)^a}$$

Cut-off regularization: $\int d^4 k_E \longrightarrow \pi^2 \int_0^{\Lambda^2} dk_E^2 k_E^2$

$$A(a; \Delta) = \frac{i(-1)^a}{16\pi^2} \int_0^{\Lambda^2} dk_E^2 \frac{k_E^2}{(k_E^2 + \Delta)^a}$$

The integral is finite and does not need regularization for $a > 2$. The case

$a = 1$ was done before. For $a = 2$:

$$\begin{aligned} A(2; \Delta) &= \frac{i}{16\pi^2} \left(\ln \frac{\Lambda^2 + \Delta}{\Delta} - \ln \frac{\Lambda^2}{\Lambda^2 + \Delta} \right) \\ &= \frac{i}{16\pi^2} \left(\ln \frac{\Lambda^2}{\Delta} - 1 + O\left(\frac{\Delta}{\Lambda^2}\right) \right) \\ &\quad \vdots \\ &\quad \text{logarithmically divergent} \end{aligned}$$

Dimensional regularization

Assume space / momentum space is d dimensional and k_E^μ is a d -dimensional vector. Replace

$$\int \frac{d^4 k_E}{(2\pi)^4} \longrightarrow \tilde{\mu}^{4-d} \int \frac{d^d k_E}{(2\pi)^d} \quad \text{with} \quad \tilde{\mu} \equiv \mu \sqrt{\frac{e^{\gamma_E}}{4\pi}} \quad \begin{array}{l} \gamma_E = 0.5772 \dots \\ \text{Euler constant} \end{array}$$

Here $\tilde{\mu}(\mu)$ is an arbitrary parameter with mass dimension 1, which must be introduced to keep the mass dimension of the original integral unchanged.

Note that $A(a; \Delta)$ is now convergent, when $d - 2a < 0$. The idea is to compute it for $d < 2a$ and then continue the result analytically (in complex d space) to $d = 4$.

$$\int \frac{d^d k_E}{(2\pi)^d} \stackrel{\text{drop subscript } E}{=} \int_0^{+\infty} dk^2 \frac{1}{2} (k^2)^{\frac{d}{2}-1} \int \frac{d\Omega^{(d)}}{(2\pi)^d}$$

$$= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Gamma(\frac{d}{2})} \int_0^{+\infty} dk^2 (k^2)^{\frac{d}{2}-1}$$

Standard trick to compute the surface of the unit sphere in d dim.

$$\begin{aligned} (\sqrt{\pi})^d &= \left(\int_0^{+\infty} dx e^{-x^2} \right)^d \\ &= \int_0^{+\infty} d^d x e^{-\sum_{i=1}^d x_i^2} \\ &= \int dx x^{d-1} e^{-x^2} \int d\Omega^{(d)} \\ &= \frac{1}{2} \Gamma\left(\frac{d}{2}\right) \int d\Omega^{(d)} \end{aligned}$$

Then

$$\begin{aligned} A(\alpha; \Delta) &= \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{(-1)^\alpha}{\Gamma(\frac{d}{2})} \mu^{4-d} \int_0^\infty \frac{d\ell \ell^{\frac{d}{2}-1}}{(\ell + \Delta)^\alpha} \quad [\ell \equiv k^2] \\ &= \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{(-1)^\alpha}{\Gamma(\frac{d}{2})} \Delta^{\frac{d}{2}-\alpha} \mu^{4-d} \int_0^1 dx x^{\alpha-1-\frac{d}{2}} (1-x)^{\frac{d}{2}-1} \\ &\quad \uparrow \\ &\quad x = \frac{\Delta}{\ell + \Delta} \text{ so } \ell = \Delta \frac{1-x}{x}, \ell + \Delta = \frac{\Delta}{x} \\ &= \frac{i}{(4\pi)^2} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \left(\frac{\Delta}{4\pi \mu^2} \right)^{\frac{d}{2}-2} (-\Delta)^{2-\alpha} \quad (*) \\ &\quad \uparrow \\ &\quad \int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)} \end{aligned}$$

Remember the following properties of the Γ -fn $\Gamma(x) = \int_0^\infty dt \, t^{x-1} e^{-t}$

- $\Gamma(1+x) = x \Gamma(x)$
- $\Gamma(n+1) = n!$ for $n = 0, 1, \dots$
- $\Gamma(x)$ is an analytic fn in the complex plane with simple poles at $x = 0, -1, -2, \dots$
- $\Gamma(1+\varepsilon) = \exp\left(-\varepsilon\gamma_E + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_n \varepsilon^n\right)$

$$\zeta_k = \sum_{m=1}^{\infty} \frac{1}{m^k}$$

Riemann zeta fn.

$A(a; \Delta)$ is singular for $d=4$ when $a=1, 2$. The original integral diverges for all $a \leq 2$, but in dimensional regularization the analytic continuation in d gives a finite value.

For $a=1, 2$ we put $d=4-2\varepsilon$ and expand around $\varepsilon=0$:

$$A(1, \Delta) = \frac{i}{(4\pi)^2} (-\Delta) \underbrace{\frac{\Gamma(-1+\varepsilon)}{\Gamma(1)}}_{(-1)\left(\frac{1}{\varepsilon} - \gamma_E + 1 + o(\varepsilon)\right)} \underbrace{\left(\frac{\Delta}{4\pi\hat{\mu}^2}\right)^{-\varepsilon}}_{\rightarrow \frac{\Delta}{\hat{\mu}^2} e^{\gamma_E \varepsilon}}$$

$$= \frac{i}{(4\pi)^2} \times \Delta \times \left[\frac{1}{\varepsilon} - \ln \frac{\Delta}{\hat{\mu}^2} + 1 + o(\varepsilon) \right]$$

Note: no γ_E , $\log 4\pi$

– this motivated the choice of $\hat{\mu}$ in the definition.

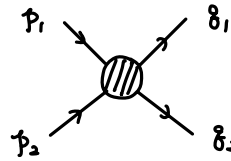
$$A(2, \Delta) = \frac{i}{(4\pi)^2} \left[\frac{1}{\varepsilon} - \ln \frac{\Delta}{\hat{\mu}^2} + o(\varepsilon) \right]$$

Comparing this result to the cut-off regularization, one notes the correspondence

$$\frac{1}{\varepsilon} - \ln \frac{\Delta}{\hat{\mu}^2} \longleftrightarrow \ln \frac{\Lambda^2}{\Delta^2}$$

$\phi\phi \rightarrow \phi\phi$ scattering at the one loop

$$\langle g_1 g_2 ; \text{out} | p_1 p_2 ; \text{in} \rangle_{\text{connected}} =$$



connected, amputated

$$= (2\pi)^4 \delta^{(4)}(g_1 + g_2 - p_1 - p_2)$$

$$\times (\sqrt{Z})^4 \hat{G}_{\text{amp}}^{(4)}(g_1, g_2, -p_1, -p_2) \Big|_{g_i^2 = p_i^2 = m^2}$$

$$= iT$$

Since $\sqrt{Z} = 1 + O(\lambda_0^2)$ so can set $\sqrt{Z}^4 \rightarrow 1$ at one loop

$$\hat{G}_{\text{amp}}^{(4)} =$$

$$+ O(\lambda_0^3)$$

Note :



must not be included, because we compute the amputated Green fn. Its effect is in Z and m

$$= -i\lambda_0 + (-i\lambda_0)^2 \times \frac{1}{2} \times \left[\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} \frac{i}{(p_1 + p_2 + k)^2 - m_0^2 + i\epsilon} + \text{two further terms with } p_1 + p_2 \rightarrow p_1 - g_1 \text{ and } p_1 + p_2 \rightarrow p_1 - g_2 \right]$$

$\uparrow$
 symmetry factor

Before continuing we remark on the mass dimension of couplings and fields in dimensional regularization.

$$e^{iS} = e^{i \int d^d x \mathcal{L}} \quad \Rightarrow \quad \begin{aligned} [\mathcal{L}] &= d \\ [\partial_\mu] &= 1 \quad [m_0] = 1 \\ [S] &= 0 \end{aligned}$$

Kinetic term $[\partial \phi_0 \partial \phi_0] = d \quad \Rightarrow \quad [\phi_0] = \frac{d}{2} - 1$

Interaction term $[\lambda_0 \phi_0^4] = d \quad \Rightarrow \quad [\lambda_0] = d - 4(\frac{d}{2} - 1) = 4 - d = 2\varepsilon$

$\Rightarrow$ The bare coupling is no longer dimensionless. In general, the dimension of fields and couplings depends on d .

Computation of the loop integral:

$$\begin{aligned} & \hat{\mu}^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m_0^2 + i\varepsilon) [(p+k)^2 - m_0^2 + i\varepsilon]} \\ &= \hat{\mu}^{4-d} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[x k^2 + \bar{x} (p+k)^2 - m_0^2 + i\varepsilon]^2} \quad (\bar{x} = 1-x) \end{aligned}$$

$$k^2 + 2\bar{x} p \cdot k + \bar{x} p^2 - m_0^2 + i\varepsilon = (k + \bar{x} p)^2 + x \bar{x} p^2 - m_0^2 + i\varepsilon$$

then change integration variable to $k' \equiv k + \bar{x} p$, and subsequently

rename $k' \rightarrow k$

$$\begin{aligned} &= \int_0^1 dx \hat{\mu}^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 - (m_0^2 - x \bar{x} p^2 - i\varepsilon)]^2} \\ &= \frac{i}{(4\pi)^2} \left[\frac{1}{\varepsilon} - \int_0^1 dx \ln \frac{m_0^2 - x \bar{x} p^2 - i\varepsilon}{\mu^2} + o(\varepsilon) \right] \\ &= \frac{i}{(4\pi)^2} \left[\underbrace{\frac{1}{\varepsilon} - \ln \frac{m_0^2}{\mu^2}}_{-\ln \frac{m_0^2}{\Lambda^2} - 1 \text{ with cut-off regularization}} - \underbrace{\int_0^1 dx \ln \left(1 - x \bar{x} \frac{p^2}{m_0^2} - i\varepsilon \right)}_{\equiv A(p^2)} + o(\varepsilon) \right] \end{aligned}$$

Hence

$$\hat{G}_{\text{amp}}^{(4)} = -i\lambda_0 \left\{ 1 - \frac{\lambda_0 \tilde{\mu}^{-2\varepsilon}}{32\pi^2} \left[\frac{3}{\varepsilon} - 3 \ln \frac{m_0^2}{\mu^2} - A(s) - A(t) - A(u) \right] + o(\lambda_0^2) \right\}$$

"s, t, u Mandelstam variables"

Divergent as $d \rightarrow 4$ ($\varepsilon \rightarrow 0$)

Also depends on the regularization procedure

Definition of a physical interaction strength through the value of the $2 \rightarrow 2$ scattering amplitude at the production threshold $s = 4m^2$, $t = u = 0$

(all three-momenta vanish):

$$(\sqrt{z})^4 \hat{G}_{\text{amp}}^{(4)} \Big|_{\substack{s=4m^2 \\ t=u=0}} = -i\lambda \tilde{\mu}^{2\varepsilon}$$

↑
this is needed in dimensional regularization

so that $[\lambda] = 0$

$$\xrightarrow[A(0)=0]{} \lambda \tilde{\mu}^{2\varepsilon} = \lambda_0 \left\{ 1 - \frac{\lambda_0 \tilde{\mu}^{-2\varepsilon}}{32\pi^2} \left[\frac{3}{\varepsilon} - 3 \ln \frac{m_0^2}{\mu^2} - A(4m^2) \right] + \dots \right\}$$

with cut-off regularization $\Rightarrow \lambda = \lambda_0 \left\{ 1 - \frac{\lambda_0}{32\pi^2} \left[3 \ln \frac{\Lambda^2}{m_0^2} - 3 - A(4m^2) \right] + \dots \right\}$

$\lambda = \lambda(\lambda_0, m_0)$ depends on the regularization, but λ can be measured directly by measuring $2 \rightarrow 2$ scattering at threshold (in practice: close to threshold).

Then express the scattering amplitude for arbitrary kinematics in terms of λ

$$\begin{aligned}
 (\sqrt{z})^4 \hat{G}_{\text{amp}}^{(4)} &= -i\lambda \hat{\mu}^{2\epsilon} \left\{ 1 + \frac{\lambda}{32\pi^2} \left[\frac{3}{\epsilon} - 3 \ln \frac{m_0^2}{\mu^2} - A(4m^2) \right] + \dots \right\} \\
 &\quad \times \left\{ 1 - \frac{\lambda}{32\pi^2} \left[\frac{3}{\epsilon} - 3 \ln \frac{m_0^2}{\mu^2} - A(s) - A(t) - A(u) \right] + \dots \right\} \\
 &= -i\lambda \left\{ 1 + \frac{\lambda}{32\pi^2} \left[A(s) + A(t) + A(u) - A(4m^2) \right] + o(\lambda^2) \right\} \\
 &\quad \uparrow \\
 &\quad \text{now can take } \epsilon \rightarrow 0, \text{ also } m_0 = m + o(\lambda)
 \end{aligned}$$

- $d\sigma/d\Omega$ is now a calculable prediction of the theory including the quantum correction.

- The above result is independent of the intermediate regularization.

This worked because the UV divergence was independent of the kinematics.

This was the case because it arises from loop momenta larger than any masses or momenta.

The definition of λ is not unique and the functional form of the scattering amplitude depends on this definition. However, its numerical value does not.

For example, could have defined some λ' by

$$\begin{aligned}
 (\sqrt{z})^4 \hat{G}_{\text{amp}}^{(4)} \Big|_{s=t=u=-4v^2} &\equiv -i\lambda' \hat{\mu}^{2\epsilon} \\
 &\quad \uparrow \\
 &\quad \text{arbitrary ; choose some value}
 \end{aligned}$$

Then

$$(\sqrt{z})^4 \hat{G}_{\text{amp}}^{(4)} = -i\lambda' \left\{ 1 + \frac{\lambda'}{32\pi^2} \left[A(s) + A(t) + A(u) - 3A(-4v^2) \right] + \dots \right\}$$

but because

$$\lambda' = \lambda \left\{ 1 + \frac{\lambda}{32\pi^2} \left[3A(-4v^2) - A(4m^2) \right] + \dots \right\}$$

the numerical value is the same if the numerical values of λ and λ' are correctly related according to the previous equation.

Renormalized perturbation theory

Renormalized fields

$$\int d^4x e^{ip \cdot x} \langle \Omega | T \{ \phi_0(x) \phi_0(0) \} | \Omega \rangle \xrightarrow{p^2 \rightarrow m^2} \frac{iZ}{p^2 - m^2 + i\epsilon} + \dots$$

Define the on-shell renormalized field $\phi_0 = \sqrt{Z} \phi$
 and Green fns ↖ renormalized field

$$G^{(n)}(x_1 \dots x_n) = \langle \Omega | T \{ \phi(x_1) \dots \phi(x_n) \} | \Omega \rangle$$

$$G^{(n)}(x_1 \dots x_n) = (\sqrt{Z})^{-n} G_0^{(n)}(x_1 \dots x_n) \quad \langle \Omega | T \{ \phi_0(x_1) \dots \phi_0(x_n) \} | \Omega \rangle$$

and

$$G_{\text{amp}}^{(n)}(p_1 \dots p_n) = (\sqrt{Z})^n G_0^{(n)} \text{amp}(p_1 \dots p_n)$$

since n 2-pt fns

are amputated and

$$G^{(2)} \sim Z^{-1} G_0^{(2)}$$

This is exactly the combination which enters
the LSZ reduction formula.

Computation of renormalized Green functions in bare perturbation theory:

- (1) Compute the relations $m(m_0, \lambda_0)$, $\lambda(m_0, \lambda_0)$, $Z(m_0, \lambda_0)$ to the required order in perturbation theory (defining λ through an observable). Any convenient regularization can be used.
- (2) Compute $G_0^{(n)}(x_1 \dots x_n)$ to the desired order in terms of λ_0, m_0
- (3) Eliminate λ_0, m_0 in the expression for $G_0^{(n)}(x_1 \dots x_n)$ in favor of λ, m and multiply by $(\sqrt{Z})^{-n}$ where $Z = Z(m, \lambda)$, $\lambda_0(m, \lambda)$. The renormalized Green functions are free divergences and independent of the regularization.

An alternative scheme is renormalized perturbation theory, which never uses explicit bare parameters. This scheme often leads to a simpler book-keeping for more complicated theories and processes, and particular in higher loop orders.

(1) Write $\phi_0 = \sqrt{Z} \phi$, $m_0^2 = m^2 + \delta m^2$, $\lambda_0 = Z_\lambda \lambda \tilde{\mu}^{2\epsilon}$ where Z , δm^2 , Z_λ still need to be determined.

(2) Since $Z - 1$, $Z_\lambda - 1$, δm^2 are (at least) $O(\lambda)$, we can express the Lagrangian in terms of the renormalized fields and parameters and treat the difference as a perturbation.

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} (\partial^\mu \phi_0 \partial_\mu \phi_0 - m_0^2 \phi_0^2) - \frac{\lambda_0}{4!} \phi_0^4 \\ &= \frac{1}{2} Z \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z (m^2 + \delta m^2) \phi^2 - Z_\lambda Z^2 \tilde{\mu}^{2\epsilon} \frac{\lambda}{4!} \phi^4 \\ &= \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda \tilde{\mu}^{2\epsilon}}{4!} \phi^4 \\ &\quad + \frac{1}{2} (Z - 1) \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} [(Z - 1) m^2 + Z \delta m^2] \phi^2 - (Z_\lambda Z^2 - 1) \frac{\lambda \tilde{\mu}^{2\epsilon}}{4!} \phi^4\end{aligned}$$

$$\equiv \mathcal{L}_r + \mathcal{L}_{ct}$$

↑
first line in previous eq.

$\mathcal{L}_r$ has the same functional

form as $\mathcal{L}$

"counterterm Lagrangian". The terms

are treated as new interactions, i.e.

$\mathcal{L}_{ct}$ is considered as part of $\mathcal{L}_{int}$

for the derivation of the Feynman rules.

(3) Compute the Green fns. In this case the Green fns are directly the renormalized ones (only ϕ appears ever). The Feynman propagator of the ϕ field is $\frac{i}{p^2 - m^2 + i\epsilon}$ [not: $m_0^2 - i\epsilon$], and the result is directly expressed in terms of m, λ [m_0, λ_0 never appear]

The counterterms $\delta_Z \equiv Z - 1$, $\delta_m \equiv (Z-1)m^2 + Z\delta m^2$, $\delta_\lambda \equiv Z_\lambda Z - 1$ have to be determined order by order in the expansion in λ by three renormalization conditions. The scheme defined above corresponds to

$$\text{exact, renormalized 2pt fn.} = \frac{i}{p^2 - m^2 - \pi(p^2, m^2)} \xrightarrow{p^2 \rightarrow m^2} \frac{i}{p^2 - m^2} \quad \begin{array}{l} \text{to all orders} \\ \text{in } \lambda \end{array}$$

↑
self-energy in renormalized
perturbation theory

$$\Rightarrow \pi(p^2, m^2) \Big|_{p^2=m^2} = 0 \qquad \frac{d\pi}{dp^2} \Big|_{p^2=m^2} = 0 \qquad (*)$$


 amputated, Fourier-transformed, $s=4m^2, t=u=0$

$$= -i\lambda \tilde{\mu}^{2\epsilon}$$
 to all order in PT

"Four-point fn in renormalized PT"

Explicitly in the one-loop order

$$-i\pi^{(1)} = \text{Feynman diagram with a loop} + \text{Feynman diagram with a cross} \quad \text{with Feynman rule } -i(\delta_Z p^2 + \delta_m)$$

$$\stackrel{\text{use cut-off regularization}}{=} \frac{-i\lambda}{32\pi^2} \left[\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} + o\left(\frac{m^4}{\Lambda^2}\right) \right] - i(\delta_Z p^2 + \delta_m)$$

$$\xrightarrow{\vec{p}^2 = m^2} \begin{cases} \delta_Z m^2 + \delta_m = -\frac{\lambda}{32\pi^2} \left[\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} + o\left(\frac{m^4}{\Lambda^2}\right) \right] \\ \delta_Z = 0 \end{cases}$$

$$\Rightarrow \delta_m \underset{\uparrow}{=} \delta m^2 = -\frac{\lambda}{32\pi^2} \left[\Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} + o\left(\frac{m^4}{\Lambda^2}\right) \right]$$

up to $O(\alpha)$

$$\begin{aligned}
 \text{Diagram with shaded circle} |_{\text{amp}} &= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} \leftarrow \text{counterterm vertex} \\
 &\stackrel{!}{=} -i\lambda\tilde{\mu}^{2\epsilon} \quad \text{determines } \delta\lambda \text{ at } O(\lambda) \quad -i\delta\lambda\lambda\tilde{\mu}^{2\epsilon}
 \end{aligned}$$

Two-loop order

$$\begin{aligned}
 -i\Pi^{(2)} &= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \quad \delta z, \delta m \text{ at } O(\lambda) \\
 &+ \text{Diagram 4} \quad \text{counterterm } \delta\lambda \text{ at } O(\lambda) + \text{Diagram 5} \quad \delta z, \delta m \text{ at } O(\lambda^2)
 \end{aligned}$$

The renormalization condition (*) then determines $\delta z, \delta m$ at $O(\lambda^2)$ from the two-loop computation of $\Pi^{(2)}$. If the scheme is to work out, the divergent left over before one adds Diagram 5 must have the correct form, namely $a \times p^2 + b$. Otherwise $\Pi^{(2)}$ could not be made finite with a renormalization of mass, coupling and field.

Renormalization schemes and scale-dependent parameters

Up to now we defined m , Z from the location and residue of the 2pt fn.

We now call these m_{phys} , $\underset{\substack{\uparrow \\ \text{on-shell}}}{Z_{\text{os}}}$

One can impose other renormalization condition than (*). This only amounts to a reparametrization of the theory.

$$\phi_o = \sqrt{Z_{\text{os}}} \phi_{\text{os}} = \sqrt{Z} \phi$$

$$m_o^2 = m_{\text{phys}}^2 + \delta m_{\text{phys}}^2 = m^2 + \delta m^2$$

$\Rightarrow m_{\text{phys}} = m_{\text{phys}}(m, \lambda)$ is a calculable relation, free of divergences

Different conventions / renormalization conditions are also referred to a different renormalization schemes

Computation of scattering matrix elements:

- Independent of the convention for the renormalized mass, $\tilde{G}_{\text{amp}}$ must always be computed for on-shell momenta $p_i^2 = m_{\text{phys}}^2$
- The Z -factor in the LSZ theorem is always the residue Z_{os} of the bare 2-pt fns. If one computes renormalized Green functions but uses ϕ , not ϕ_{os} , one needs to multiply $\tilde{G}_{\text{amp}}$ by $\left(\sqrt{\frac{Z_{\text{os}}}{Z}}\right)^n$ to get the scattering matrix element. Note $Z_{\text{os}} / Z = 1 + o(\lambda)$ is free from divergences.

The $\overline{MS}$ (modified minimal subtraction) scheme.

This has become the most widely used scheme. Instead of imposing renormalization conditions on certain Green functions one simply defines the counterterms δ_i such that they contain only the divergent contribution $\propto 1/\epsilon$ in dimensional regularization.

The parameters $\bar{m}, \bar{\lambda}$ in the $\overline{MS}$ scheme have no direct physical interpretation.

This does not matter as long as all observables can be related to them in an unambiguous divergent-free way.

$\overline{MS}$ mass

$$-i\pi = \frac{-i\lambda}{32\pi^2} (-m^2) \left(\frac{1}{\epsilon} - \ln \frac{m^2}{\mu^2} + 1 \right) - i(\delta_Z p^2 + \delta_m)$$

$$\text{on-shell : } \delta_m = \frac{\lambda}{32\pi^2} m^2 \left(\frac{1}{\epsilon} - \ln \frac{m^2}{\mu^2} + 1 \right) \quad [\text{so that } \pi|_{p^2=m^2} = 0]$$

$$\overline{MS} : \delta_m = \frac{\lambda}{32\pi^2} m^2 \frac{1}{\epsilon}$$

$$\Rightarrow m_0^2 = m_{\text{phys}}^2 + \delta m_{\text{phys}}^2 = m_{\text{phys}}^2 \left[1 + \frac{\lambda}{32\pi^2} \left(\frac{1}{\epsilon} - \ln \frac{m_{\text{phys}}^2}{\mu^2} + 1 \right) \right]$$

$$= \bar{m}^2(\mu) + \delta \bar{m}^2(\mu) = \bar{m}^2(\mu) \left[1 + \frac{\lambda}{32\pi^2} \left(\frac{1}{\epsilon} \right) + \dots \right]$$

$$\Rightarrow m_{\text{phys}}^2 = \bar{m}^2(\mu) \left[1 + \frac{\lambda}{32\pi^2} \left(\ln \frac{\bar{m}^2(\mu)}{\mu^2} + 1 \right) + o(\lambda^2) \right]$$

↑
independent
of μ

The $\overline{MS}$ mass depends on μ in such a way that the left-hand side is independent of μ .

"Relation between the $\overline{MS}$ and the physical (on-shell) mass"

μ can in principle be chosen arbitrarily but some choices are wiser than others, as will be seen.

$\overline{MS}$ coupling

$$\text{Diagram} \big|_{\text{amp}} = -i\lambda \tilde{\mu}^{2\epsilon} \left[1 - \frac{\lambda}{32\pi^2} \left(\frac{3}{\epsilon} - 3 \ln \frac{m^2}{\mu^2} - A(s) - A(t) - A(u) \right) + \delta\lambda \right]$$

Previously : $\delta\lambda = \frac{\lambda}{32\pi^2} \left[\frac{3}{\epsilon} - 3 \ln \frac{m^2}{\mu^2} - A(4m^2) \right]$

$\overline{MS}$: $\delta\lambda = \frac{\lambda}{32\pi^2} \left[\frac{3}{\epsilon} \right]$

$$\Rightarrow \lambda = \frac{Z_{\bar{\lambda}}}{Z_{\lambda}} \bar{\lambda}(\mu) = (1 + \delta_{\bar{\lambda}} - \delta_{\lambda} + \dots) \bar{\lambda}(\mu) \quad [\text{since } \delta_Z = 0 \text{ at } O(\lambda)]$$

$$= \bar{\lambda}(\mu) \left[1 + \frac{\bar{\lambda}(\mu)}{32\pi^2} \left(3 \ln \frac{m^2}{\mu^2} + A(4m^2) \right) + O(\bar{\lambda}^2) \right]$$

$\vdots$

scale-dependent coupling, such that the left-hand-side is independent of μ .

This is a general feature of the $\overline{MS}$ scheme. All renormalized objects depend on the scale μ introduced with the d -dim integration measure. However, one can define scale-dependent quantities also in other schemes. The usefulness of scale-dependent parameters, in particular couplings, will become apparent soon.

4.2 The systematics of renormalization

This deals with the question whether all Green functions of a given theory can be rendered finite by renormalization of the fields, mass and couplings. This is non-trivial, because there is only a finite number of parameters that can be renormalized and the structure of the divergences must match the terms that can be added consistently to the Lagrangian. We will not treat this mathematically exhaustively here. However, the main ideas will become clear.

The systematic analysis will provide us with a better understanding of the structure of quantum corrections and lead to the important concept of effective quantum field theories.

Degree of divergence and classification of renormalizable interactions

The following considerations are general and do not assume any specific Lagrangian.

The theory may contain several different fields, labelled by f

Notation:

σ : 1PI Feynmann diagram

L : number of loops

I_f : number of internal lines (propagators) of field of type f

E_f : number of external lines (propagators) of field of type f

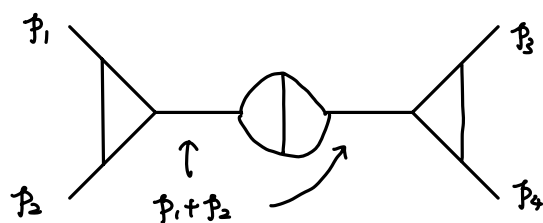
V_i : number of vertices of type i

a_i : number of derivatives on fields in vertex of type i

n_{if} : number of fields of type f in vertex of type i

d : number of space - time dimensions

Note: it is enough to consider 1PI diagrams. In a 1PR diagram, the line that connects 1PI subgraphs does not contain any loop momentum, so the analysis can be done for every 1PI component separately.



Consider 1PI self-energy and vertex subgraphs only

Assume that all loop momenta become large at the same rate. More precisely, let

$$D(\gamma) = \max_{\text{all terms}} \left\{ \begin{array}{l} \text{powers of loop momentum in the numerator of the Feynman} \\ \text{diagram, including the integration measure} \end{array} \right.$$

$$- \left. \begin{array}{l} \text{powers of loop momentum in the} \\ \text{denominator} \end{array} \right\} \quad \begin{array}{l} \text{superficial degree of} \\ \text{divergence of } \gamma \end{array}$$

Example :

$$\int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{1}{k_1^2} \frac{1}{(k_1 - k_2)^2} \left(-g_{\mu\nu} + \frac{k_{2\mu} k_{2\nu}}{m^2} \right) \frac{1}{[(k_1 + p)^2 - M^2]^2}$$

$$\begin{array}{ccccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 4 & 4 & -2 & -2 & 0 & 2 & -4 \end{array}$$

$$\hookrightarrow D(\gamma) = \max \{ 0, 2 \} = 2 \quad \text{"quadratically divergent"}$$

The diagram will certainly be divergent if $D(\gamma) = 0$. The case $D(\gamma) = 0$ is called logarithmically divergent, since $\int^\Lambda \frac{dx}{x} \sim \ln \Lambda$

Computation of the superficial degree of divergence.

Propagator of field of type $f \sim 1/(k^2)^{1-S_f}$ for $k^2 \gg m^2$. $S_f = 0$ for the scalar field. We keep S_f general, since $S_f \neq 0$ for other field types. However, one can assume $S_f \geq 0$

$$D(\gamma) = \sum_f I_f (2S_f - 2) + \sum_i V_i q_i + d_L \quad (*)$$

For connected diagram the number of loops is

$$L = \sum_f I_f - \sum_i V_i + 1$$

overall momentum conservation
δ-fn always left over.

Also

$$2 I_f + E_f = \sum_i V_i n_{if}$$

"Every internal (external) line contributes two (one) ends which have to be tied up with an internal vertex i . The vertices supply $\sum_i V_i n_{if}$ ends."

Eliminate L, I_f in (*)

$$D(x) = d - \sum_f E_f \left(\frac{d}{2} - 1 + S_f \right) - \sum_i V_i \Delta_i$$

with $\Delta_i \equiv d - a_i - \sum_f n_{if} \left(\frac{d}{2} - 1 + S_f \right)$ is a characteristic quantity for each vertex of type i

From now on restrict to $d = 4$

Case $\Delta_i \geq 0$ for all vertices i of the theory

$$\Rightarrow D(x) \leq 4 - \sum_f E_f (1 + S_f)$$

The degree of divergence decreases when the number of external legs increase.

Only a finite number of 1PI Green functions can be divergent.

For scalar field theory

	Green function	maximal value of $D(x)$
$\delta m^2, Z$	$\phi \phi$	2
Z_{λ_2}	$\phi \phi \phi$	1
Z_{λ_4}	$\phi \phi \phi \phi$	0

This suggests that all divergences can be removed by counterterms corresponding to fields, mass and coupling renormalization of the theory

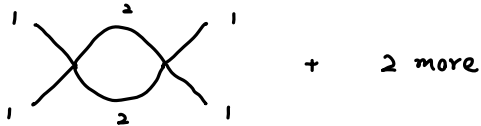
$$\mathcal{L}_0 = \frac{1}{2} (\partial^\mu \phi_0 \partial_\mu \phi_0 - m_0^2 \phi_0^2) - \frac{\lambda_3}{3!} \phi^3 - \frac{\lambda_4}{4!} \phi_0^4$$

Note sometimes one needs fewer counterterms than this analysis suggests: For $\mathcal{L} = \frac{1}{2} (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2) - \frac{\lambda_4}{4!} \phi^4$ there is no divergent three-point function, because the symmetry $\phi \rightarrow -\phi$ forbids non-zero Green functions with an odd number of fields.

The converse also occurs - one needs more counterterms than the original lagrangian suggests. Suppose we assume a theory of two scalar fields.

$$\mathcal{L} = \sum_{i=1,2} \frac{1}{2} (\partial^\mu \phi_i \partial_\mu \phi_i - m_i^2 \phi_i^2) - \frac{\lambda}{4!} \phi_1^2 \phi_2^2$$

Then we can have $\phi_1 \phi_1 \rightarrow \phi_1 \phi_1$ scattering



We know that we need a counterterm of the form $\delta\lambda_i \phi_i^4$ to make this finite, but we can get it from $\mathcal{L}$ only if we assume there is a $-\frac{\lambda_i}{4!} \phi_i^4$ terms from the beginning. Similarly for ϕ_2^4 . Hence we must add these terms to $\mathcal{L}$ and determine the value of the three couplings $\lambda, \lambda_1, \lambda_2$ by measuring the strength of $\phi_1 \phi_1 \rightarrow \phi_1 \phi_1$, $\phi_2 \phi_2 \rightarrow \phi_2 \phi_2$, $\phi_1 \phi_1 \rightarrow \phi_2 \phi_2$ scattering independently. Note that the ϕ_1^4, ϕ_2^4 terms are not forbidden by any symmetry of the original lagrangian. Hence we conclude:

A theory can be rendered finite by a reparametrisation of fields and masses and a finite number of couplings (interactions)

if a) $\Delta_i \geq 0$ for all vertices

b) $\mathcal{L}$ contains all vertices compatible with the symmetries of the (regularized) theory. (The regularization may break a symmetry.

Then it is no longer a symmetry of $\mathcal{L}$.)

Such theories are called renormalizable. Despite the divergences in Feynman diagrams these theories are fully predictive. All divergences are absorbed into parameters, which can be obtained from a finite number of measurements. Even if there were no divergences, one would have to perform precisely the same measurement to determine the parameters (masses, couplings) of the theory.

An interaction is called renormalizable if $\Delta_i = 0$ and superrenormalizable if $\Delta_i > 0$. In the latter case $D(x)$ decreases (due to $-\sum_i \gamma_i \Delta_i$) when the number of vertices of type i increases. Ultimately $D(x) < 0$ always if V_i is large enough. Hence if all $\Delta_i > 0$ the loop diagrams are finite from a certain order and there is only a finite number of divergent diagrams. Such theories are called superrenormalizable. The only such theory in $d=4$ is

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2) - \frac{\lambda}{3!} \phi^3$$

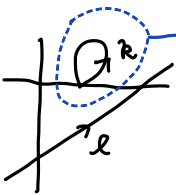
This theory is not interesting as a physical theory, because it has no stable ground state, since the potential is unbounded from below.

Case $\Delta_i < 0$ for at least one vertex type i — such theories are called non-renormalizable. We discuss them separately below.

Subdivergences and the renormalization of Feynman diagrams with several loops.

The previous analysis is incomplete, because we assumed that a diagram is convergent if $D(\gamma) < 0$. This is not always correct, since the loop momenta may become large at different rates, or only a subset may become large. In this case $D(\gamma)$ may be negative, but a subdiagram may divergent.

Example :



A Feynman diagram P consisting of a triangle with an internal loop. One of the internal lines is a loop labeled Q^k , which is circled in blue and labeled "divergent subdiagram γ ". The external lines are labeled l .

$P =$

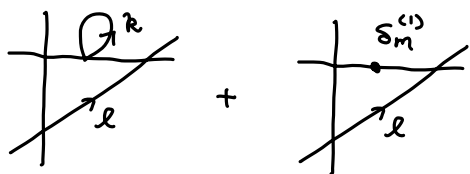
$D(P) = -2 \quad \text{but} \quad D(\gamma) = 2$

divergent as $k \rightarrow \infty$ for fixed l .

Theorem (Weinberg) : P is convergent, if $D(\gamma) < 0$ for $\gamma = P$ and all subdiagrams γ of P

— The statement appears obvious, the proof is nevertheless tedious

For the above example it is clear that the divergent subdiagram does not invalidate the previous statements about renormalizability. Work with renormalized perturbation theory and $\mathcal{L}_r + \mathcal{L}_{ct}$. Then there is a counterterm for the divergent subgraph and



Two Feynman diagrams are shown separated by a plus sign. The first diagram is the same as the one in the example, with a loop labeled Q^k . The second diagram is a triangle with an internal line labeled $\delta_m^{(1)}$ and external lines labeled l .

is finite

A more complicated situation arises if $D(P) \geq 0$ and if there are divergent subdiagrams.

Consider the example ["fake example"]

(two-loop self-energy in ϕ^4 theory)

$$\text{diagram} \equiv \Gamma$$

The computation of this diagram in dimensional regularization in the massive theory gives

$$\lambda^2 \times \left\{ p^2 [\dots] - m^2 \left[\frac{a}{\varepsilon^2} + \frac{2a}{\varepsilon} \ln \frac{\mu^2}{p^2} + \frac{b}{\varepsilon} + \text{finite} \right] \right\}$$

It is not possible to compensate the divergent part in bracket by a mass

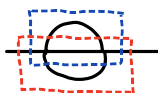
counterterm

$$\frac{\times}{\delta m^{(2)}}$$

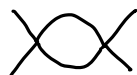
since the counterterm can never contain $\ln p^2$.

Since the Lagrangian is local, counterterms can only be polynomial in momentum

However, Γ has two divergent one-loop subdiagrams:



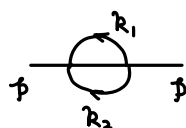
$\Rightarrow$



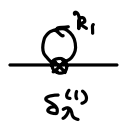
for this we have the counterterm

$$\times \delta \lambda^{(1)} \text{ in } \mathcal{L}_{ct}$$

Hence at $\mathcal{O}(\lambda^2)$ one should compute



+



region $k_2 \gg k_1$

$\vdots$

These diagrams are
of the form

$$\frac{1}{\varepsilon} \times \left(\frac{1}{\varepsilon} + \ln \frac{\mu^2}{p^2} \right)$$

from the counterterm

[from $\mathcal{O}$]

in such a way that the sum is

$$\lambda^2 \left\{ p^2 [\dots] - m^2 \left[\frac{a'}{\varepsilon^2} + \frac{b'}{\varepsilon} + \text{finite} \right] \right\}$$

actually $\frac{1}{\varepsilon} + \ln \frac{\mu^2}{m^2}$ in ϕ^4 theory.

so the left-over divergence is local and defines $\delta m^{(2)}$

The above example and statements can be generalized and proven for arbitrarily complicated Feynman diagrams (completed by work by Zimmermann and others 1970). The combinations of renormalization of all subdivergences by their corresponding counterdiagrams leads precisely to the multiplicative renormalization contained in the reparameterization of fields, masses and couplings.

Non-renormalizable and effective quantum field theories

We return to the case that $\Delta_i < 0$ for one (or more) interaction terms in $\mathcal{L}_{\text{int}}$. Then

$$D(x) = d - \sum_f E_f \left(\frac{d}{2} - 1 + S_f \right) - \underbrace{\sum_i V_i \Delta_i}_{\text{contains one positive term, hence}}$$

any Green function (any E_f) becomes divergent, if only V_i is sufficiently large.

$\Rightarrow$ Must add an infinite number of terms with more and more fields to $\mathcal{L}_{\text{int}}$ and hence an infinite number of unknown coupling constants (subject only to the symmetries of the theory), which must be determined by an infinite number of measurements.

It therefore appears that non-renormalizable theories are unpredictable. This was indeed the attitude in the early period of QFT (until about the late 1970s). The modern perspective, however, is quite different.

Consider the example $\mathcal{L} = \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2) - \frac{\lambda_4}{4!} \phi^4 - \frac{\lambda_6}{6!} \phi^6$

Since $[\phi] = 1$, the coupling has dimension $[\lambda_6] = -2$. We can write $\lambda_6 = \hat{\lambda}_6 / M^2$ where M is some scale chosen so that $\hat{\lambda}_6$ is $\mathcal{O}(1)$.

This is general. Since $[L] = d$ the dimension of a coupling g_i multiplying an interaction term with vertex V_i is

$$d = [g_i] + \Delta_i + \underbrace{\sum_f n_{if} \left(\frac{d}{2} - 1 + S_f \right)}_{\text{dimension of } [V_f]}$$

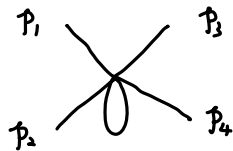
From the kinetic term in L

$$\text{always } d = 2[V_f] + 2(1 - S_f)$$

$$\Rightarrow [g_i] = \Delta_i$$

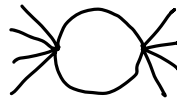
$$\text{Hence write } g_i = \frac{\hat{g}_i}{M^{-\Delta_i}}$$

Go back to the example : the ϕ^6 vertex induces $2 \rightarrow 2$ and $4 \rightarrow 4$ scattering at the one-loop level.



$$\sim \frac{\hat{\lambda}_6}{M^2} \times p^2$$

[actually $p^2 \rightarrow m^2$ here]



$$\sim \frac{\hat{\lambda}_6^2}{M^4}$$

Both loops are divergent.

The first contributes to $\delta\lambda_4$, the second requires that one adds a $-\frac{\lambda_8}{8!}\phi^8$ term.

This illustrates that above statement that one must add all possible interaction terms to L .

Now consider a scattering process where the particles involved have momentum not larger than some energy E (in the cms frame). Then an interaction with $\Delta_i < 0$ contributes an amount.

$$|T_{\beta\alpha}| \supset \hat{g}_i \frac{E^D}{M^{|\Delta_i|}} \quad \text{where } D = |\Delta_i| \text{ is the dimension of } |T_{\beta\alpha}|$$

$\Rightarrow$ If $E \ll M$ the contribution of non-renormalizable interactions to a scattering process is suppressed, the more the higher the dimension of the interaction operator.

Hence, if one is satisfied with making predictions of finite accuracy, only a finite number of non-renormalizable interactions is ever relevant. Hence, non-renormalizable QFTs are as predictive as renormalizable theories, as long as $E \ll M$. If one wants to increase the accuracy, one needs to include higher dimension operators, but the required number is always finite.

To summarize:

renormalizable QFTs : The interaction operators have dimension ≤ 4 such that $A_i \geq 0$.

The theory can in principle be used to calculate processes at arbitrary energy scales E . $\rightarrow$ It is a candidate for a fundamental / final theory of Nature.

non-renormalizable QFTs: Includes interaction operators of arbitrary dimension but only a finite number is relevant if energies are small enough $\rightarrow$ Predictive QFT for phenomena below a certain scale M . Such theories are also called effective field theories for $E \ll M$.

The distinction is in fact not as fundamental as it appears. Since one will never be able to perform experiments at arbitrarily large energies we will never know whether a renormalizable theory that works for energy E is in fact only the leading term of a non-renormalizable theory for $E \ll M$, because the two cannot be distinguished

An effective field theory example.

EFTs can be very useful in practice, even if the description in terms of a renormalizable theory exists in principle.

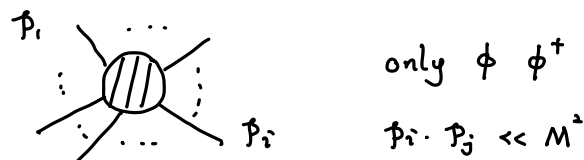
This is illustrated with the following example of a theory with two scalar fields with very different masses.

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi + \frac{1}{2} \partial_\mu \mathcal{F} \partial^\mu \mathcal{F} - \frac{1}{2} M^2 \mathcal{F}^2 - \frac{\kappa_3}{3!} \mathcal{F}^3 - g \mathcal{F} \phi^\dagger \phi$$

$$" \kappa_3, g \sim o(M) "$$

Note that this is a superrenormalizable theory. We could add the $(\phi^\dagger \phi)^2$, $\mathcal{F}^4$ and $\mathcal{F}^2 \phi^\dagger \phi$ interactions. But for the present purpose we can set them to zero.

Assume that the energy of scattering processes is such that $p_i, p_j, m^2 \ll M^2$ for any external momenta p_i, p_j . Then we only have to consider external lines of $\phi, \phi^\dagger$



Since $\mathcal{F}$ particles cannot be produced, we can write down a non-renormalizable effective theory containing only the ϕ fields, which reproduces the scattering amplitudes of the full theory to a given accuracy. One says that one "integrates out" the heavy scalar field. Thus, we need to find $\mathcal{L}_{\text{eff}}$ such that

$$T_{\beta\alpha} \Big|_{\substack{\text{computed} \\ \text{with } \mathcal{L} \\ \uparrow \\ \mathcal{L}(\phi, \mathcal{F})}} = T_{\beta\alpha} \Big|_{\substack{\text{computed} \\ \text{with } \mathcal{L}_{\text{eff}} \\ \uparrow \\ \mathcal{L}_{\text{eff}}(\phi)}} + o\left(\left(\frac{E}{M}\right)^q\right)$$

depends on desired accuracy of the approximation.

Consider $\phi\phi \rightarrow \phi\phi$

$$\begin{array}{c} p_1 \\ \diagdown \\ \text{---} \\ \diagup \\ p_2 \end{array} \quad \begin{array}{c} p_3 \\ \diagup \\ \text{---} \\ \diagdown \\ p_4 \end{array} + \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} = (-ig)^2 \left[\frac{i}{t-M^2} + \frac{i}{u-M^2} \right]$$

$$\approx (+i) \frac{2g^2}{M^2} + o\left(g^2 \frac{E^2}{M^4}\right) = \begin{array}{c} \diagup \\ \text{---} \\ \diagdown \end{array} + \dots$$

We can reproduce this by writing

$$\mathcal{L}_{\text{eff}}^{(4)} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda_4}{4} (\phi^\dagger \phi)^2 \quad \text{where } \lambda_4 = -2g^2/M^2.$$

But we need more vertices:

$$\begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} + \dots \quad (3 \text{ more})$$

is not completely reproduced by

$\mathcal{L}_{\text{eff}}$, which contains only

$$\begin{array}{c} \text{---} \\ \diagdown \\ \text{---} \\ \diagup \\ \text{---} \end{array} + 3 \text{ more}$$

$$\begin{array}{c} \vdots \\ (-ig)^3 (-ik_3) \frac{(-i)^3}{M^6} \Rightarrow \frac{g^3 k_3}{M^6} (\phi^\dagger \phi)^6 \end{array}$$

$$\text{So we need } \mathcal{L}^{(6)} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda_4}{4} (\phi^\dagger \phi)^2 - \frac{\lambda_6}{8M^2} (\phi^\dagger \phi)^3$$

$$\text{with } \lambda_6 \sim g^3 k_3 / M^4 \text{ dimensionless}$$

At this order in $1/M^2$ one should also include a term $(\phi^\dagger \partial_\mu \phi)(\partial^\mu \phi^\dagger \phi)$ which

comes from the $O(g^2 p^2/M^4)$ correction to $\phi\phi \rightarrow \phi\phi$ scattering. Thus it is clear

that we can replace

$$\mathcal{L} \rightarrow \mathcal{L}_{\text{eff}}$$

for the purpose of describing low-energy ($E \ll M$) ϕ scattering processes.

This extends to loop diagrams :

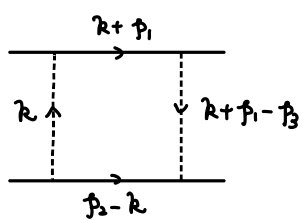
$$\text{Diagram with } f \text{ loop} = \frac{\text{Diagram with } \lambda_4 \sim g^2}{\lambda_4 \sim g^2} + \frac{\text{Diagram with } Z, \delta m^2 \sim g^2}{Z, \delta m^2 \sim g^2} + \dots$$

which determines the change of the mass and field renormalization due to the presence of f

$$\text{Diagram} = \text{Diagram with } \lambda_4 + \text{Diagram with } \lambda_4$$

a new contribution to λ_4 of order g^4/m^4

The successive computation of corrections to the couplings of the EFT is called matching. For this to work it is important that remainders are local. i.e. arise from loop momentum $k \sim M \gg E$. This always work out, because by construction $\mathcal{L}_{\text{eff}}$ reproduces the low-energy physics from $k \sim E$. To see this consider the above diagram



$k \sim O(p_i)$

can replace $\frac{1}{k^2 - M^2}$, $\frac{1}{(k + p_1 - p_2)^2 - M^2}$ by $-\frac{1}{M^2}$

This corresponds to

This contribution is a complicated fn of the p_i and not polynomial (= local)

$k \sim O(M)$

can expand in all p_i so the integral is

$$\int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4 (k^2 - M^2)^2} \times \text{polynomial in } p_i$$

which can be reproduced by a local vertex.



We can take two perspectives on this example:

- (1) If the full theory is known we can use the EFT, because the calculations are simpler — provided we are interested only in physics at scales $E \ll M$. The simplicity arises because the effective couplings $\lambda_4, \lambda_6, \dots$ are process-independent and can be determined once and forever by matching. Afterwards one can use the simpler theory of a single scalar field.
- (2) If no experiment ever accessed energy scales of $O(M)$ we cannot know for certain what is the full theory. All accessible physics can be described by a non-renormalizable theory of a single scalar field, where the couplings $\lambda_4, \lambda_6, \dots$ are determined by experiment. Once their values are known, this provides constraints on possible full theories at higher scales, because the measured value of λ_6 may not be consistent with the one predicted by the hypothesized full theory.

Renormalization of composite operators.

It is sometimes necessary to consider scattering matrix elements (and hence Green fns) with insertions of composite operators $O[\phi](x) = \text{product of fields at the same space-time point (local) operators, possibly with derivatives.}$ But Green functions such as

$$\langle \Omega | T \{ O[\phi](x) \phi(y_1) \phi(y_2) \dots \} | \Omega \rangle$$

(or more complicated ones with several composite operators) are not rendered finite after renormalization of fields, masses and the coupling parameters of the Lagrangian.

Example: ϕ^4 theory, operator ϕ^2

$$\text{let } \phi_0 = \sqrt{Z} \phi, \quad \phi_0^2 = Z \phi^2 \phi^2$$

Consider the renormalized Green function

$$G(x, y_1, y_2) = \langle \Omega | T \{ \phi^2(x) \phi(y_1) \phi(y_2) \} | \Omega \rangle$$

$$= (Z\phi^2)^{-1} (\sqrt{Z})^{-2} \langle \Omega | T \{ \phi_0^2(x) \phi_0(y_1) \phi_0(y_2) \} | \Omega \rangle$$

$$= \begin{array}{ccccccc} \text{diagram (1)} & + & \text{diagram (2)} & + & \text{diagram (3)} & + [(Z-1) + (Z\phi^2-1)] \text{diagram (4)} & + O(\lambda^2) \end{array}$$

(1) (2) (3) (4)

The divergence in these two diagrams is cancelled by replacing $m_0^2 \rightarrow m_0^2(m, \lambda)$ in the two propagators in



The naive assumption would be $Z\phi^2 = (\sqrt{Z})^2$ from the renormalization of the fields.

But in ϕ^4 theory $\sqrt{Z} = 1 + O(\lambda^2)$, hence contribution (4) vanishes at $O(\lambda)$, But (2) has a logarithmic divergence, so the renormalized Green function would not be finite!

↳ Need operator renormalization, that is $Z\phi^2 \neq (\sqrt{Z})^2$

Then choose $Z\phi^2 - 1$ such that (4) cancels the divergent part of (2).

In the $\overline{MS}$ scheme $Z\phi^2 - 1 = \frac{1}{\epsilon}$ pole part of (2)

The reason why $\phi_a^2(x)$ renormalizes differently from $\phi_a(x)\phi_a(y)$ no matter what $y-x$ is that the local limit $y \rightarrow x$ generates new ultraviolet divergences.

There is in fact a close relation between composite operator and coupling

renormalization. Add

$\mathcal{L} \supset C_0 \overset{\text{--- composite operator}}{\underbrace{O_0(x)}} \quad \text{to the effective Lagrangian}$
 $\quad \quad \quad \uparrow$
 $\quad \quad \quad \text{"coupling"}$

[e.g. $\phi^3, \phi^4, \phi^6 \dots$]

The fact that one needs coupling renormalization, i.e. $C_0 = Z_C C$ is related to the fact that rescaling the fields in O_0 is not enough to make Green fns finite.

In case of Lagrangian terms it is conventional to associate the Z -factor with the coupling C , in case of "external" operators one associates it with the operator itself.

Operator mixing

Suppose O is a local composite operator of mass dimension D . e.g. ϕ^4 in ϕ^4 -theory. Then $O_0 = Z_0 O$ may not be enough to make all Green fns with elementary fields and an insertion of O finite, if there are more operators of the same dimension.

Example,

$$\langle \Omega | T \{ \phi_0^4(x) \phi_0(y_1) \phi_0(y_2) \} | \Omega \rangle = \text{diagram: a horizontal line with points } y_1, x, y_2 \text{ and a loop above } x$$

$$\supset \frac{1}{\epsilon} (a p^2 + b m^2) \quad \text{divergent part} \propto p^2, m^2 \text{ by dimensional reasons}$$

Fourier transform Here $a = 0$

This divergence cannot be cancelled by any rescaling $Z\phi^4 = 1 + \dots$ or counterterm from $\mathcal{L}$ simply because there is no tree diagram, but the divergent part has the structure of the operator $\partial^\mu \phi \partial_\mu \phi$ ($\leftarrow a$) or $m^2 \phi^2$ ($\leftarrow b$) corresponding to $\text{---}x\text{---} \quad a p^2 + b m^2$

The solution is operator mixing. One needs to consider all operators of dimension

D , here $\phi^4, m^2 \phi^2, \partial_\mu \phi \partial^\mu \phi, m^4 \mathbb{1}$

Then

$$O_{i0} = Z_{ij} O_j$$

$\uparrow$
matrix of renormalization constants

$$\Rightarrow \langle \Omega | T \{ \phi^4(x) \phi(y_1) \phi(y_2) \} | \Omega \rangle = (Z^{-1})_{1j} Z^{-1} \langle \Omega | T \{ O_j(x) \phi(y_1) \phi(y_2) \} | \Omega \rangle$$

$$= \frac{\text{loop}}{\phi_0^4} + \frac{\text{tadpole}}{m_0^2 Z_{12}^{-1}} + \frac{\text{tadpole}}{(p^2) Z_{13}^{-1}} + \dots$$

$$\stackrel{!}{=} \text{finite}$$

$$\Rightarrow (Z^{-1})_{13} = 0 \quad \text{since no } p^2/\epsilon \quad \text{and} \quad Z_{13}^{-1} \neq 0$$

In this way divergences can always be removed by renormalizing the complete set of local operators. Simplifications apply to operators that vanish by the equation of motion. Such as $\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2$.

4.3 The renormalization group

Begin the discussion with two observables

- The need for regularization always introduces a new scale Λ for cut-off regularization

μ (or $\tilde{\mu}$) for dimensional regularization. Note that μ should not be viewed as a cut-off. The analogue of the cut-off is $1/\epsilon$, but one needs the renormalization scale μ for dimensional reasons.

When $\Lambda \rightarrow \infty$ or $\epsilon \rightarrow 0$ the dependence on the scale may remain depending on the renormalization condition. Such as ν in the definition of $\lambda'(\nu)$ or the dim. reg. scale μ in the $\overline{MS}$ scheme $\bar{\lambda}(\mu)$

If $\mathcal{L}$ was scale-invariant as a classical theory ($\mathcal{L}$ contains no dimensionful parameters, in particular mass), scale-invariance is certainly broken in the regularized theory. When $\Lambda \rightarrow \infty$ or $\epsilon \rightarrow 0$ scale invariance is usually not recovered, hence dilatation symmetry is anomalous.

- Consider $\phi\phi \rightarrow \phi\phi$ scattering at high energies, $s \sim -t \sim -u \gg m^2$ for the case where the coupling λ was defined by

$$(\sqrt{2})^4 \tilde{G}_{\text{amp}}^{(4)} \Big|_{\substack{s=4m^2 \\ t=u=0}} = -i\lambda$$

$$(2\lambda - 1) = \frac{\lambda}{32\pi^2} \left[\frac{3}{\epsilon} - 3\mathcal{L}_1 \frac{m^2}{\mu^2} - A(4m^2) \right]$$

Then

$$i T_{\phi\phi \rightarrow \phi\phi} = -i\lambda \left\{ 1 + \frac{\lambda}{32\pi^2} [A(s) + A(t) + A(u)] + o(\lambda^2) \right\}$$

$$\xrightarrow{s, |t|, |u| \gg m^2} -i\lambda \left\{ 1 + \frac{\lambda}{32\pi^2} \left[\ln \frac{-s}{m^2} + \ln \frac{-t}{m^2} + \ln \frac{-u}{m^2} \right] + \dots \right\}$$

Since $A(p^2) = \int_0^1 dx \ln \frac{m^2 - x\bar{x}p^2}{m^2} \xrightarrow{p^2 \gg m^2} \ln \frac{-p^2}{m^2}$

$\Rightarrow$ The perturbation expansion breaks down even for small λ when $\lambda \ln \frac{s}{m^2} \sim o(1)$,
i.e. in the high-energy limit.

This is a strange result: one would have expected that one can simply compute the process setting $m=0$. Instead the result diverges for $m \rightarrow 0$ and the expansion breaks down.

The reason for this unexpected behaviour is that the renormalization condition, which defines Z_λ , depends on m^2 and does not allow the limit $m \rightarrow 0$. To investigate the high-energy limit one must use a definition of λ , which does not have this property, for instance the $\overline{MS}$ coupling $\bar{\lambda}(\mu)$ or the coupling $\lambda'(\nu)$.

Use $\overline{MS}$:

$$Z_\lambda^{\overline{MS}} = 1 + \frac{\lambda(\mu)}{32\pi^2} \frac{3}{\epsilon} + o(\lambda^2)$$

$$\lambda(\mu)/\lambda = Z_\lambda / Z_\lambda^{\overline{MS}} = 1 - \frac{\lambda}{32\pi^2} \left[3 \ln \frac{m^2}{\mu^2} - A(4m^2) \right] + o(\lambda^2) \quad (*)$$

$$\Rightarrow i T_{\phi\phi \rightarrow \phi\phi} = -i\lambda(\mu) \left\{ 1 + \frac{\lambda(\mu)}{32\pi^2} \left[\ln \frac{-s}{\mu^2} + \ln \frac{-t}{\mu^2} + \ln \frac{-u}{\mu^2} \right] + o(\lambda^2) \right\}$$

There are no large logarithms for $s, -t, -u \gg m^2$ provided one chooses the renormalization scale $\mu \sim o(\sqrt{s})$ and perturbation theory is well behaved provided $\lambda(\mu) \ll 1$ for $\mu \sim o(\sqrt{s})$, i.e. provided the scale-dependent or "running" coupling is small at the characteristic scale of the process.

Computation of the scale dependence.

Question is how to compute $\lambda(\mu)$ for different μ .

Suppose λ is known. Cannot use (*) to obtain $\lambda(\sqrt{s})$ directly, because the expansion in $\lambda \ln \frac{\mu^2}{s} \sim O(1)$ is not well-defined. Instead

$$\begin{array}{ccccc} \lambda & \xrightarrow{(*)} & \lambda(\mu_1) & \xrightarrow{\quad} & \lambda(\mu_2) \\ & & \uparrow & & \\ & & \text{evolve the MS coupling from } \mu_1 \text{ to } \mu_2 & & \end{array}$$

Bare coupling λ_0 is independent of μ , $\lambda_0 = Z_\lambda^{\overline{MS}} \mu^{2\epsilon} \lambda(\mu)$

$$\begin{aligned} \Rightarrow \mu^2 \frac{d}{d\mu^2} \lambda(\mu) &= \mu^2 \frac{d}{d\mu^2} \left[\hat{\mu}^{-2\epsilon} (Z_\lambda^{\overline{MS}})^{-1} \right] \lambda_0 \\ &= -\epsilon \lambda(\mu) - \frac{1}{Z_\lambda^{\overline{MS}}} \left(\mu^2 \frac{d}{d\mu^2} Z_\lambda^{\overline{MS}} \right) \lambda(\mu) \quad (+) \\ &\equiv \beta(\lambda) \quad [\text{finite as } \epsilon \rightarrow 0] \quad \text{"Simplest form of a renormalization group equation"} \end{aligned}$$

$$\Rightarrow \frac{d\mu^2}{\mu^2} = \frac{d\lambda}{\beta(\lambda)} \quad \Rightarrow \quad \ln \frac{\mu_2^2}{\mu_1^2} = \int_{\lambda(\mu_1)}^{\lambda(\mu_2)} \frac{d\lambda}{\beta(\lambda)}$$

This allows to compute $\lambda(\mu_2)$ from $\lambda(\mu_1)$, given μ_1 and μ_2 .

Similarly for masses, Write $m_0 = Z_m m(\mu)$.

$$\begin{array}{ccccc} \mu^2 \frac{d}{d\mu^2} m(\mu) &= & \mu^2 \frac{d}{d\mu^2} (Z_m^{-1}) m_0 &= & \underbrace{\left(-\frac{1}{Z_m} \mu^2 \frac{d}{d\mu^2} Z_m \right)}_{\equiv \gamma_m(\lambda)} m(\mu) \\ & & \uparrow & & \\ & & m_0 \text{ independent of } \mu & & \end{array}$$

$$\Rightarrow \frac{dm}{m(\mu)} = \frac{d\mu^2}{\mu^2} \gamma_m(\lambda) = d\lambda \frac{\gamma_m(\lambda)}{\beta(\lambda)}$$

$$\Rightarrow \ln \frac{m(\mu_2)}{m(\mu_1)} = \int_{\lambda(\mu_1)}^{\lambda(\mu_2)} d\lambda \frac{\gamma_m(\lambda)}{\beta(\lambda)}$$

$$\Rightarrow m(\mu_2) = m(\mu_1) \exp \left[\int_{\mu_1}^{\mu_2} d\lambda \frac{\gamma_m(\lambda)}{\beta(\lambda)} \right]$$

From this can obtain $m(\mu_2)$ from $m(\mu_1)$. For simplicity assumed that $\mathcal{L}$ contains

only one coupling. Otherwise, one has a coupled system of several differential eqns

4.3.1 Overview of possible scale dependence

A) Trivial infrared (IR) fixed point.

This case is realized in ϕ^4 theory discussed above and corresponds to

$\beta(\lambda) > 0$ for small λ (and $\epsilon = 0$)

$$\phi^4 \text{ theory: } Z_\lambda = 1 + \frac{3\lambda}{32\pi^2\epsilon}$$

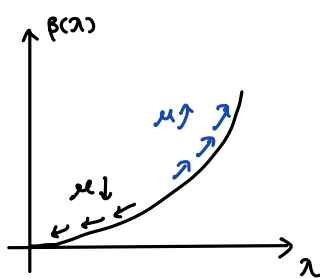
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 drop superscript $\overline{\text{MS}}$

$$\Rightarrow -\frac{1}{Z_\lambda} \frac{d}{d\ln\mu^2} Z_\lambda = -\frac{3}{32\pi^2\epsilon} \frac{d}{d\ln\mu^2} \lambda(\mu) + o(\lambda^2)$$

$$\beta(\lambda) = -\epsilon\lambda + o(\lambda^2) \Rightarrow \frac{3}{32\pi^2} \lambda(\mu) + o(\lambda^2)$$

$$\text{i.e. } \beta(\lambda) = -\epsilon\lambda(\mu) + \frac{3}{32\pi^2} \lambda^2(\mu) + o(\lambda^3)$$

$$\stackrel{\epsilon \rightarrow 0}{=} \frac{3}{32\pi^2} \lambda^2 + o(\lambda^3) \equiv \beta_0 \lambda^2 + \dots$$



$$\begin{aligned} \ln \frac{\mu_2^2}{\mu_1^2} &= \int_{\lambda(\mu_1)}^{\lambda(\mu_2)} d\lambda \left[\frac{1}{\beta_0 \lambda^2} + o\left(\frac{1}{\lambda}\right) \right] \\ &= \frac{1}{\beta_0} \left[\frac{1}{\lambda(\mu_1)} - \frac{1}{\lambda(\mu_2)} \right] + \dots \end{aligned}$$

$$\Rightarrow \lambda(\mu_2) = \frac{\lambda(\mu_1)}{1 - \beta_0 \lambda(\mu_1) \ln \frac{\mu_2^2}{\mu_1^2}}$$

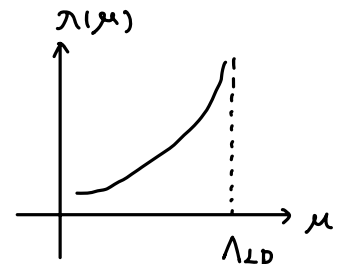
Hence choosing $\mu_1 \sim o(\mu)$ can compute $\lambda(\mu_2)$ for $\mu_2 \sim O(\sqrt{s})$. Here we only need that $\lambda(\mu)$ is small in the interval $\mu \in [\mu_1, \mu_2]$ but not that $\lambda \ln \mu_2/\mu_1$ is small!

Since $\beta_0 > 0$ $\lambda(\mu)$ increase with μ and diverges at

$$\Lambda_{LP}^2 = \mu^2 e^{\frac{1}{\beta_0 \lambda(\mu)}}$$

the Landau pole scale.

While for $\mu \rightarrow 0$ also $\lambda(\mu) \rightarrow 0$ (IR fixed point)



- Since $\lambda(\mu)$ needs to be small for the above to be valid, the question what really happens near Λ_{LP} can not be answered in perturbation theory. One would need all terms in the expansion of $\beta(\lambda)$.

However, one can conclude that the interaction becomes strong and non-perturbative at high energy. Note that if $\lambda(\mu)$ is small at low energies Λ_{LP} is exponentially large.

- The term "fixed point" refers to the fact that $\beta(\lambda_*) = 0$ for $\lambda_* = 0$. At the fixed point, the running of λ stops. Trivial fixed points are those with vanishing values of couplings, since then the theory is effectively free. Here this is the case for $\mu \rightarrow 0$ (IR)
- The use of running couplings can be viewed as a resummation of certain large terms in the perturbative expansion to all orders.

Let $q^2 \approx s \approx -t \approx -u$, then

$$i T \phi\phi \rightarrow \phi\phi = -i\lambda(q) \left[1 + \frac{\lambda(q)}{32\pi^2} \times \begin{array}{l} \text{coefficient of } \text{ocv}, \\ \text{no large logs} \end{array} + \dots \right]$$

Inserting $\lambda(q) = \frac{\lambda(m)}{1 - \frac{3}{32\pi^2} \lambda(m) \ln \frac{q^2}{m^2}}$ one recovers the starting expression

$$i T \phi\phi \rightarrow \phi\phi = -i\lambda \left\{ 1 + \frac{\lambda}{32\pi^2} [A(s) + A(t) + A(u)] + o(\lambda^2) \right\}$$

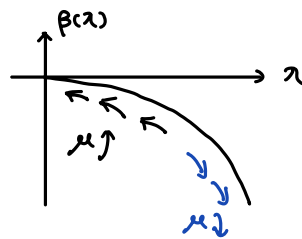
$$\xrightarrow{s, |t|, |u| \gg m^2} -i\lambda \left\{ 1 + \frac{\lambda}{32\pi^2} \left[\ln \frac{-s}{m^2} + \ln \frac{-t}{m^2} + \ln \frac{-u}{m^2} \right] + \dots \right\}$$

and an infinite series of terms $\lambda^{n+1}(m) \ln^n \frac{q^2}{m^2}$ ("leading logarithms" in every order). The running coupling sums these terms systematically.

B) Trivial ultraviolet (UV) fixed point:

$$\beta(\lambda) = \beta_0 \lambda^2 + \dots \quad \text{but } \beta_0 < 0$$

$$\lambda(\mu_2) = \frac{\lambda(\mu_1)}{1 + (-\beta_0) \lambda(\mu_1) \ln \frac{\mu_2^2}{\mu_1^2}}$$



The coupling decreases with $\mu \uparrow$ i.e. effective interaction strength becomes weaker for higher energy scattering ($\rightarrow$ trivial UV fixed point)

On the other hand, as μ decreases λ increases and becomes infinite at some scale Λ_{IR} . Again this should be interpreted as entering a region of strong coupling, not accessible to perturbation theory.

This occurs for the theory of the strong interaction, quantum chromodynamics.

For experiment.

$$\alpha_s^{\overline{MS}}(m_Z) = \frac{[g_s^{\overline{MS}}(m_Z)]^2}{4\pi} \approx 0.118$$

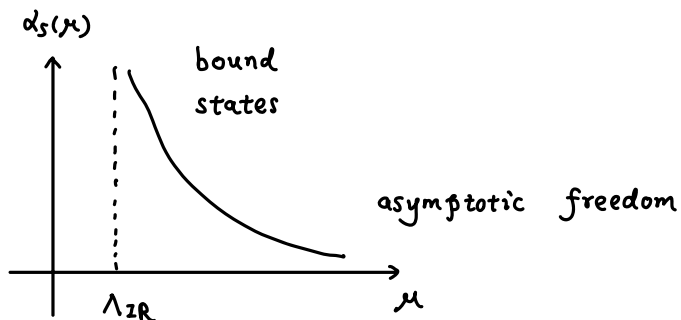
$\uparrow$
mass of the Z^0 boson $\approx 91 \text{ GeV}$

$$\alpha_s(\mu) \xrightarrow{\mu \rightarrow \infty} 0$$

"asymptotically free"

$$\alpha_s(\mu) \rightarrow \infty \text{ for}$$

$$\Lambda_{IR} \sim 200 \text{ MeV}$$

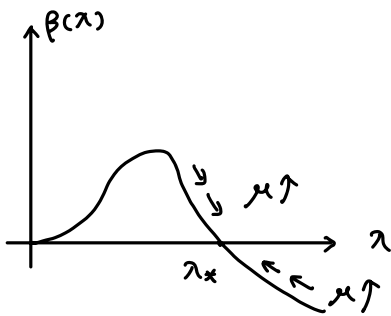


C) Non-trivial fixed points

This refers to situations where there exists a $\lambda_* \neq 0$ such that $\beta(\lambda_*) = 0$.

Two cases:

(I):



UV stable fixed point:

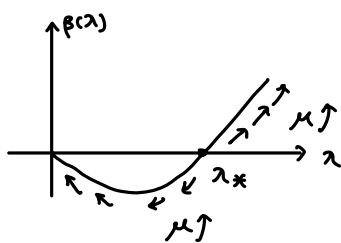
i.e. independent of the starting value λ

$$\lambda \xrightarrow{\mu \rightarrow \infty} \lambda_*$$

This occurs since for $\lambda < \lambda_*$ $\beta(\lambda) > 0$ hence increasing μ increases λ until λ_* is reached. For $\lambda > \lambda_*$ $\beta(\lambda) < 0$ and λ decreases as μ increases.

For $\mu \rightarrow 0$ the behavior depends on whether one starts with $\lambda > \lambda_*$ or $\lambda < \lambda_*$. In the above example $\lambda = 0$ is an IR stable, trivial fixed point.

(II)



IR stable fixed point:

For μ increasing, λ is repelled from the fixed point.

In the example $\lambda = 0$ is a UV-stable, trivial fixed point.

Renormalization group flow.

This can be generalized to theories with several coupling constants λ_i . We assume first $[\lambda_i] = \Delta_i = 0$ (renormalizable).

$$\mu^2 \frac{d}{d\mu^2} \lambda_i(\mu) = \beta_i(\lambda_j) = \beta_i^{jk} \lambda_j \lambda_k + \dots$$

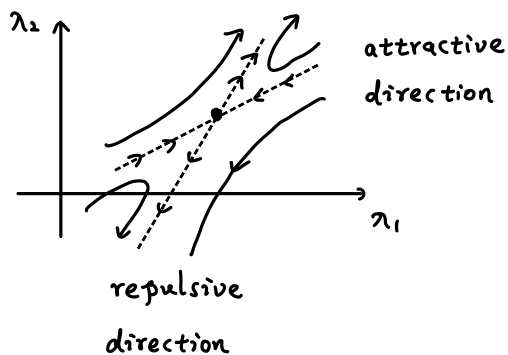
Coupled system of differential equations

Fixed point solutions: a point λ_j^* in the space of coupling constants, where $\beta_i(\lambda_j^*) = 0 \quad \forall i$ is a fixed point, since $\lambda_j(\mu) = \lambda_j^*$ is a solution.

Fixed points can be attractive or repulsive. Depends on the eigenvalues of the linearized system.

$$\mu^2 \frac{d}{d\mu^2} \lambda_i = \underbrace{\left[\frac{\partial \beta_i}{\partial \lambda_k} \Big|_{\lambda^*} \right]}_{\text{constant matrix } A_{ik}} (\lambda_k - \lambda_k^*)$$

Attractive in the direction of negative eigenvalues, repulsive for positive ones.



Flow indicated for increasing μ

Now include dimensionful coupling constants and consider the flow of couplings in a non-renormalizable theory.

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \sum_i \lambda_i \mathcal{O}_i \\ &= \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2(\mu) \phi^2 - \frac{\lambda_4(\mu)}{4!} \phi^4 - \frac{\lambda_6(\mu)}{6!} \phi^6 + \dots \\ &\quad + \mathcal{L}_{ct}\end{aligned}$$

Introduce dimensionless couplings : $\hat{\lambda}_2 = \frac{m^2(\mu)}{\mu^2}$, $\hat{\lambda}_4 = \lambda_4(\mu)$, $\hat{\lambda}_6 = \lambda_6(\mu) \mu^2$ etc.

Then

$$\mu^2 \frac{d}{d\mu^2} \hat{\lambda}_i = \beta_i(\hat{\lambda}_j) = -\frac{\Delta_i}{2} \hat{\lambda}_i + \beta_i^{jk} \hat{\lambda}_j \hat{\lambda}_k + \dots$$

since for small $\hat{\lambda}_i$ the μ -dependence is determined by the trivial factor $(\mu^2)^{-\frac{\Delta_i}{2}}$

Now assume the theory is valid at a large scale $\Lambda \gg$ energy of scattering experiments and masses. Also assume that at this scale all couplings $\hat{\lambda}_i$ are of the same order $\mathcal{O}(1)$ but small enough to allow a perturbative expansion.

Then

$$\frac{d\hat{\lambda}_i}{\hat{\lambda}_i} = -\frac{\Delta_i}{2} \frac{d\mu^2}{\mu^2} \quad \rightarrow \quad \ln \hat{\lambda}_i = -\frac{\Delta_i}{2} \ln \mu^2 + \text{const.}$$

$$\Rightarrow \quad \hat{\lambda}_i(\mu) = \hat{\lambda}_i(\Lambda) \times \left(\frac{\mu}{\Lambda} \right)^{-\Delta_i} + \text{corrections.}$$

- Non-renormalizable couplings ($\Delta_i < 0$) become unimportant for $\mu \ll \Lambda$, namely of order $\left(\frac{\mu}{\Lambda} \right)^{|\Delta_i|}$ in agreement with the previous discussion.

This explains why low-energy phenomena are described approximately by renormalizable QFTs.

Operators $\mathcal{O}_i$ with $[\mathcal{O}_i] > 4$ are therefore also called irrelevant.

- Superrenormalizable couplings grow for $\mu \ll \Lambda$. This applies in particular for mass terms:

$$m^2(\mu) = \mu^2 \hat{\lambda}_2(\mu) = \Lambda^2 \hat{\lambda}_2(\Lambda) = O(\Lambda^2)$$

A mass term $m^2(\mu) \ll \Lambda^2$ requires that one fine-tunes $\hat{\lambda}_2(\Lambda)$ to very small values.

Operators with $[O_i] < 4$ are called relevant.

- Renormalizable couplings and operators with $[O_i] = 4$ are called marginal.

Whether they grow or not for $\mu \downarrow$ is determined by the next term, β_i^{jk} , not by their mass dimension.

4.3.3 Renormalization group equation for Green functions

Renormalization group functions:

$$\beta_i(\lambda_j) = \mu^2 \frac{d}{d\mu^2} \lambda_i$$

$$\gamma_m(\lambda_j) = \frac{\mu^2}{m} \frac{\partial m}{\partial \mu^2} \quad \text{mass anomalous dimension} \quad (+)$$

$$\gamma_\phi(\lambda_j) = \frac{\mu^2}{Z_\phi} \frac{\partial Z_\phi}{\partial \mu^2} \quad \text{field anomalous dimension.}$$

For composite operators

$$\gamma_{mn}(\lambda_j) = (Z^{-1})_{mk} \mu^2 \frac{d}{d\mu^2} Z_{kn} \quad \begin{array}{l} \text{operator anomalous} \\ \text{dimension matrix.} \end{array}$$

The origin of the term "anomalous dimension" will become apparent.

Now derive the scale-dependence of n-pt fns.

mass dimension denoted by d_G

$$\tilde{G}^{(n)}(p_i, m, \lambda)$$

renormalized n-pt fn expressed in terms of
renormalized parameters.

(momentum space)

Now rescale all external momenta by a common factor: $p_i \rightarrow a p_i$

$$p_i/p_j \rightarrow p_i/p_j \quad p_i/m \rightarrow a p_i/m = p_i/(m/a)$$

Hence

$$\tilde{G}^{(n)}(a p_i; m, \lambda) = a^{d_G} \tilde{G}^{(n)}(p_i; m/a, \lambda)$$

But this wrong

because $\tilde{G}^{(n)}$ also depends explicitly on the renormalization scale.

The correct statement is

$$\tilde{G}^{(n)}(a p_i; m, \lambda; \mu) = a^{d_G} \tilde{G}^{(n)}(p_i; \frac{m}{a}, \lambda; \frac{\mu}{a}) \quad (*)$$

The goal is to construct a RG eqn, that allows us to change μ/a back to μ . Note m, λ on both side of (*) refers to $m(\mu), \lambda(\mu)$.

The bare Green function is independent of μ :

$$\mu^2 \frac{d}{d\mu^2} \left[Z_\phi^{1/2} \tilde{G}^{(n)}(p_i; m(\mu), \lambda(\mu); \mu) \right] = 0$$

$$\Rightarrow \left(\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \lambda}{\partial \mu^2} \frac{\partial}{\partial \lambda} + \mu^2 \frac{\partial m}{\partial \mu^2} \frac{\partial}{\partial m} + \frac{n}{2} \frac{\mu^2}{Z_\phi} \frac{\partial Z_\phi}{\partial \mu^2} \right) \tilde{G}^{(n)}(p_i; m(\mu), \lambda(\mu); \mu)$$

$$= \left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\lambda) \frac{\partial}{\partial \lambda} + \gamma_m(\lambda) m \frac{\partial}{\partial m} + \frac{n}{2} \gamma_\phi(\lambda) \right] \tilde{G}^{(n)}(p_i; m(\mu), \lambda(\mu); \mu)$$

$$= 0 \quad \quad \quad \uparrow$$

Define this as $F(\mu)$, where $m(\mu), \lambda(\mu)$

are solutions to (*) with initial condition

$$m(\mu_0) = m, \quad \lambda(\mu_0) = \lambda.$$

Then

$$\left[\mu^2 \frac{d}{d\mu^2} + \frac{n}{2} \gamma_\phi(\lambda(\mu)) \right] F(\mu) = 0$$

$$\Rightarrow F(\mu) = F(\mu_0) \exp \left[-\frac{n}{2} \int_{\mu_0^2}^{\mu^2} \frac{dv^2}{v^2} \gamma_\phi(\lambda(v)) \right]$$

Explicitly

$$\tilde{G}^{(n)}(p_i; m(\mu), \lambda(\mu); \mu) = \tilde{G}^{(n)}(p_i; m(\mu_0), \lambda(\mu_0); \mu_0) e^{-\frac{n}{2} \int_{\mu_0^2}^{\mu^2} \frac{dv^2}{v^2} \gamma_\phi(\lambda(v))}$$

Use this in (*) [left-hand side] with $p_i \rightarrow a p_i$

$$\tilde{G}^{(n)}(a p_i; m(\mu_0), \lambda(\mu_0); \mu_0) e^{-\frac{n}{2} \int_{\mu_0^2}^{\mu^2} \frac{dv^2}{v^2} \gamma_\phi(\lambda(v))} = a^{d_G} \tilde{G}^{(n)}(p_i; \frac{m(\mu)}{a}, \lambda(\mu); \frac{\mu}{a})$$

Now choose $\mu = a\mu_0$, then rename $\mu_0 \rightarrow \mu$

$$\hat{G}^{(n)}(ap_i; m(\mu), \lambda(\mu); \mu) = a^{d_G} \exp \left[\int_1^a \frac{da'}{a'} n \gamma_\phi(\lambda(a'\mu)) \right] \times \hat{G}^{(n)}(p_i; \frac{m(a\mu)}{a}, \lambda(a\mu); \mu) \quad (**)$$

Comparing this to the naive, wrong result, this is an additional factor, which modifies the naive dimensional scaling a^{d_G} , involving the anomalous dimension of the field and a change in the mass and coupling arguments, which takes into account the scale dependence of the coupling:

Notes:

(1) For amputated Green fns $\mu^2 \frac{d}{d\mu^2} [Z_\phi^{-n/2} \tilde{G}_{\text{amp}}^{(n)}(\dots)] = 0$, hence

$$n \gamma_\phi \rightarrow -n \gamma_\phi \text{ in } (**)$$

(2) If the n-pt fn involves a composite operator $O_i(x)$ instead of $\phi(x)$, there is a contribution from the operator anomalous dimension.

Application & Examples.

Consider the 2pt function of a scalar field $\hat{G}^{(2)}(p^2; m(\mu), \lambda(\mu); \mu)$

$$d_G = -2$$

Take $a = p/p_0$

$$\hat{G}^{(2)}(p^2; m(\mu), \lambda(\mu); \mu) = \frac{p_0^2}{p^2} e^{\int_1^a \frac{da'}{a'} 2\gamma_\phi(\lambda(a'\mu))} \hat{G}^{(2)}(p_0^2; \frac{m(a\mu)}{a}, g(a\mu); \mu)$$

Case a) small coupling $\gamma_\phi(\lambda) = \gamma_0 \lambda + \dots$

$$\beta(\lambda) = \beta_0 \lambda^2 + \dots$$

$$\exp\left[\int_1^a \frac{da'}{a'} 2\gamma_\phi(\lambda(a'\mu))\right] = \exp\left[\int_{\mu^2}^{a^2\mu^2} \frac{d\mu'^2}{\mu'^2} \gamma_\phi(\lambda(\mu'))\right]$$

$$= \exp\left[\int_{\lambda(\mu)}^{\lambda(a\mu)} d\lambda' \frac{\gamma_\phi(\lambda')}{\beta(\lambda')}\right]$$

$$\approx \exp\left[\int_{\lambda(\mu)}^{\lambda(a\mu)} d\lambda' \frac{\gamma_0}{\beta_0 \lambda'}\right]$$

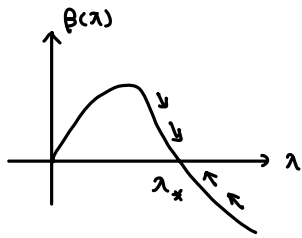
$$= \exp\left[\frac{\gamma_0}{\beta_0} \ln \frac{\lambda(a\mu)}{\lambda(\mu)}\right] = \left[\frac{\lambda(a\mu)}{\lambda(\mu)}\right]^{\gamma_0/\beta_0}$$

$$\Rightarrow \hat{G}^{(2)}(p^2; m(\mu), \lambda(\mu); \mu) \underset{p^2 \rightarrow \infty}{\propto} \frac{1}{p^2} \lambda(p)^{\gamma_0/\beta_0} \times \left[p\text{-independent} + o(\lambda(p), m^2/p^2) \right]$$

↑
Deviation from the naive scaling

behavior $1/p^2$ from dimensional analysis

Case b) Non-trivial fixed point.



For $\alpha \rightarrow \infty$, $\lambda \rightarrow \lambda_*$

Hence

$$\begin{aligned} \exp \left[\int_1^{\alpha} \frac{d\alpha'}{\alpha'} 2\delta\phi(\lambda(\alpha'; \mu)) \right] &\approx \exp \left(2\delta\phi(\lambda_*) \int_1^{\alpha} \frac{d\alpha'}{\alpha'} \right) \\ &= \left(\frac{p^2}{p_0^2} \right)^{\delta\phi(\lambda_*)} \end{aligned}$$

$$\Rightarrow \hat{G}^{(2)}(p^2; m(\mu), \lambda(\mu); \mu) \underset{p^2 \rightarrow \infty}{\propto} \frac{1}{(p^2)^{1 - \delta\phi(\lambda_*)}} \times [\text{p-independent}]$$

↑

Compared to the small-coupling logarithmic modification, here the power-law changes and in this sense the field dimension changes from $1 \rightarrow 1 - \delta\phi(\lambda_*)$ near the fixed point.

The example shows that near a fixed-point quantum fields can behave like objects with a dimension different from their canonical mass dimension. This can have important implications for renormalization theory. Our analysis of renormalizable and non-renormalizable theories was based on perturbation theory. If the theory has a non-trivial UV fixed point. (a question that cannot be answered in the context of a perturbative expansion in λ , which assumes $\lambda \rightarrow 0$ while λ_* is a finite number), a theory which looks non-renormalizable by perturbative power-counting may in fact be renormalizable. In this case what counts is the anomalous dimensions at the fixed point.