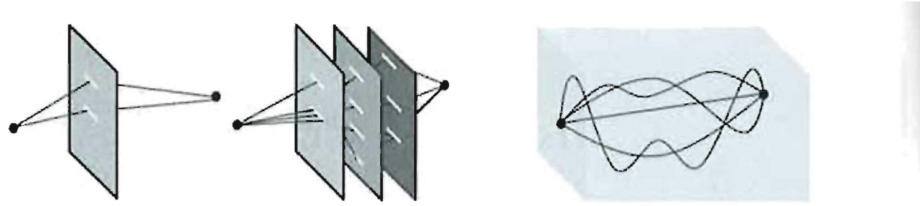
3. Path integrals & Functional methods.



The classic double slit allows for two paths between the initial and final points. Adding more screens and more slits allows for more diverse paths. An infinite number of screens and slits makes the amplitude the sum over all possible paths, as encapsulated in the path integral.

3.1 The path integral in quantum mechanics

Consider a QM system with only one coordinate q and time independent Hamiltonian $H(p, q)$

$$\text{position operator}: \begin{cases} Q_H(t) & \text{Heisenberg picture} \\ Q_S & \text{Schrödinger} \cdots \end{cases}$$

with $Q_H(t) = e^{iHt/\hbar} \underset{\substack{\uparrow \\ \text{taken at a fixed} \\ \text{time (say, } t=0)}}{Q_S} e^{-iHt/\hbar}$ [I keep $\hbar$ to highlight the relation of classical to quantum quantities]

$Q_S(t) = Q_S(0) = Q_S$, $|q, t\rangle_S = |q, 0\rangle_S = |q\rangle_S$ doesn't depend on t

$|\psi, t\rangle_S$ depends on t

$|q, t\rangle_H$, $Q_H(t)$ depends on t

$|\psi, t\rangle_H = |\psi, 0\rangle_H = |\psi\rangle_H$ doesn't depend on t

Choose $Q_H(0) = Q_S(0) = Q_S$, $|\psi, 0\rangle_S = |\psi, 0\rangle_H = |\psi\rangle_H$

$$|\psi, t\rangle_S = e^{-iH(P_S, Q_S)t/\hbar} |\psi, 0\rangle_S = e^{-iH(P_S, Q_S)t/\hbar} |\psi\rangle_H$$

$\boxed{\text{Feynman kernel}}$

The path integral is formulated to describe the time evolution of matrix elements between eigenstates of the $Q_H(t)$

Let $|q,t\rangle_H$ denotes such an eigenstate,

$$Q_H(t)\,|q,t\rangle_H = q\,|q,t\rangle_H$$ ⟵ use lower case q and p to denote eigenvalues

Here $|q,t\rangle_H$ is related to the Schrödinger eigenstates $|q\rangle_s$ of Q_s by

$$|q\rangle_s = e^{-iHt/\hbar}\,|q,t\rangle_H$$

see :
$$\begin{aligned} Q_H(t)\,|q,t\rangle_H &= e^{iHt/\hbar}\,Q_s\,e^{-iHt/\hbar}\,|q,t\rangle_H \\ &= e^{iHt/\hbar}\,Q_s\,|q\rangle_s \\ &= q\,e^{iHt/\hbar}\,|q\rangle_s \\ &= q\,|q,t\rangle_H \end{aligned}$$

1° $|q,t\rangle_H$ is the time-dependent state, $|\psi\rangle_H$ is time-independent.

2° $|q,t\rangle_H$ is not an eigenstate of $Q_H(t')$ for $t'\neq t$

3° $|q,t\rangle_H$ are complete $\displaystyle\int_{-\infty}^{+\infty} dq\,|q,t\rangle_H\,{}_H\langle q,t| = e^{iHt/\hbar}\left(\int_{-\infty}^{+\infty} dq\,|q\rangle_s\,{}_s\langle q|\right)e^{-iHt/\hbar}$

$= 1$

Consider the inner product between two Heisenberg states

Feynman kernel $\quad U(q'', t' ; q', t') = {}_H\langle q'', t'' | q', t' \rangle_H$

Rewrite U in terms of Schrödinger - picture states

$$U(q'', t'' ; q', t') = {}_s\langle q'' | e^{-iH(t''-t')/\hbar} | q' \rangle_s$$

Then U governs the evolution of Schrödinger - picture wave functions

The path integral

Rewrite the exponential in U as an infinite product:

$$U(q'', t'' ; q', t') = \lim_{n\to\infty} \prod_{i=1}^{n} \langle q'' | (1 - iH\delta t/\hbar)^n | q' \rangle$$

with $\delta t = [(t''-t')/n] \to 0$ in the limit.

Insert the identity operator $\quad I = \int dq \, |q\rangle_s \, {}_s\langle q|$

$$U(q'', t'' ; q', t') = \lim_{n\to\infty} \left[\prod_{i=1}^{n} \int dq_i \langle q_i | (1 - iH\delta t/\hbar) | q_{i-1} \rangle \, \delta(q_n - q'') \right]$$

with $q_0 \equiv q'$.

E.g. $\quad n = 2$

$\langle q'' | q_2 \rangle \langle q_2 | \quad | q_1 \rangle \langle q_1 | \quad | q' \rangle \equiv | q_0 \rangle$

$t'' = t_2 \qquad t_1 \qquad t' = t_0$

$\leftarrow$ time

$$U(q'', t'' ; q', t') = \int dq_1 dq_2 \, \langle q'' | q_2 \rangle \langle q_2 | (1 - iH\delta t/\hbar) | q_1 \rangle$$

$$\times \langle q_1 | (1 - i\delta t \, H/\hbar) | q' \rangle$$

$$= \prod_{i=1}^{2} \delta(q'' - q_2) \, \langle q_i | (1 - iH\delta t/\hbar) | q_{i-1} \rangle$$

Reduction to classical quantities

Insert unity in the form of a complete set of conjugate momentum eigenstates.

$$I = \int dp \; |p\rangle\langle p|$$

$$U(q'', t''; q', t') = \lim_{n\to\infty} \left[(2\pi\hbar)^{-n/2} \prod_{i=1}^{n-1} \int dq_i \, dp_i \int dp_0 \exp(i p_0 q_1/\hbar) \right.$$

$$\times \langle p_i | (1 - iH\delta t/\hbar) | q_i \rangle \exp(i p_i q_{i+1}/\hbar) \langle p_0 | (1 - iH\delta t/\hbar) | q' \rangle$$

where $\quad \langle q_i | p_{i-1} \rangle = (2\pi\hbar)^{-1/2} e^{i p_{i-1} q_i/\hbar}$

E.g. $\quad n = 2$

$$\langle q'' | q_2 \rangle \langle q_2 | p_1 \rangle \langle p_1 | \quad |q_1\rangle \langle q_1 | p_0 \rangle \langle p_0 | \quad |q'\rangle \equiv |q_0\rangle$$

$t'' = t_2 \qquad\qquad t_1 \qquad\qquad t'$

$$U(q'', t''; q', t') = \int dq_1 \, dp_1 \int dp_0 \; \langle q_2 | p_1 \rangle \langle p_1 | (1 - iH\delta t/\hbar) | q_1 \rangle$$

$$\times \langle q_1 | p_0 \rangle \langle p_0 | (1 - iH\delta t/\hbar) | q_0 \rangle$$

$$= (2\pi\hbar)^{-1} \int dq_1 \, dp_1 \int dp_0 \; e^{i p_0 q_1/\hbar} \langle p_0 | (1 - iH\delta t/\hbar) | q_0 \rangle$$

$$\times \; e^{i p_1 q_2/\hbar} \langle p_1 | (1 - iH\delta t/\hbar) | q_1 \rangle$$

If the Hamiltonian has the form $\qquad H(P, Q) = P^2/2m + V(Q)$

then the matrix element is

$$\langle p_i | (1 - iH(P,Q)\delta t/\hbar) | q_i \rangle = (2\pi\hbar)^{-1/2} [1 - ih(p_i, q_i)\delta t/\hbar] \, e^{-i p_i q_i/\hbar}$$

the Feynman kernel

$$U(q'', t''; q', t') = \lim_{n\to\infty} \left\{ (2\pi\hbar)^{-n} \prod_{i=1}^{n-1} \int dq_i \, dp_i \, dp_0 \; e^{i p_0 (q_1 - q_0)/\hbar} \right.$$

$$\times [1 - ih(p_i, q_i)\delta t/\hbar] \exp[i p_i (q_{i+1} - q_i)/\hbar]$$

$$\left. \times [1 - ih(p_0, q_0)\delta t/\hbar] \right\}$$

going back from the infinite product to an exponential

$$\lim_{n\to\infty} \left\{ \prod_{i=1}^{n-1} \left[1 - i h(p_i, q_i) \frac{(t''-t')}{n\hbar} \right] \right\} = \exp\left\{ \frac{-i(t''-t')}{n\hbar} \sum_{i=1}^{n-1} h(p_i, q_i) \right\}$$

$$U(q'', t''; q', t') = \lim_{n\to\infty} \left\{ (2\pi\hbar)^{-n} \prod_{i=1}^{n-1} \int dq_i \, dp_i \, dp_0 \right.$$

$$\left. \times \exp\left(\frac{i}{\hbar} \sum_{j=0}^{n-1} \delta t \left[p_j \frac{q_{j+1} - q_j}{\delta t} - h(p_j, q_j) \right] \right) \right\}$$

$$\equiv \int [dp][dq] \exp\left(\frac{i}{\hbar} \int_{t'}^{t''} dt \left[p(t) \dot{q}(t) - H(p(t), q(t)) \right] \right) \qquad (*)$$

the sum over paths, including normalization

Comments :

1° $p(t)$, $q(t)$ are mere variables of integration , not constrained to obey e. o. f.

2° It is difficult to use $(*)$ as a source of rigorous theorems.

more rigorous : Euclidean space $\to$ Minkowski space.

We use the Minkowski space formulation in perturbation theory

Configuration space:

Consider a Hamiltonian like $H = P^2/2m + V(Q)$

The momentum integrals are all Gaussian

$$\int_{-\infty}^{+\infty} dp \, \exp(-\alpha p^2 + \beta p) = \left(\frac{\pi}{\alpha}\right)^{1/2} \exp(\beta^2/4\alpha)$$

with $\alpha = i\,\delta t/(2m\hbar)$, $\beta = i(q_{j+1} - q_j)/\hbar$

so
$$U(q'', t''; q', t') = \lim_{n\to\infty} \left[\left(\frac{m}{2\pi i\hbar\delta t}\right)^{n/2} \prod_{i=1}^{n-1} \int dq_i \, \exp\left(\frac{i}{\hbar}\sum_{j=0}^{n-1} \delta t \left\{ \frac{m}{2}\left[\frac{q_{j+1}-q_j}{\delta t}\right]^2 - V(q_j) \right\}\right)\right]$$
$$= \lim_{n\to\infty} \left[\left(\frac{m}{2\pi i\hbar\delta t}\right)^{n/2} \prod_{i=1}^{n-1} \int dq_i \, \exp\left(\frac{i}{\hbar}\sum_{j=0}^{n-1} \delta t \, L(q_j, \dot{q}_j)\right)\right]$$

Lagrangian of the theory

configuration space path integral.

$$U(q'', t''; q', t') = \int_{q'}^{q''} [dq] \, \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} dt \, L(q(t), \dot{q}(t))\right]$$

$$= \int [dq] \, \exp\left[\frac{i}{\hbar} S(q'', t''; q', t')\right]$$

- 仅仅是一个形式上的表达式，一般的情形下并不能证明这个极限是存在的.
- The transition amplitude involves all possible trajectories weighted with the phase factor e^{iS}

Time-ordered products

A time-ordered product of coordinate operators

$$ {}_H\langle q'', t'' | T[Q_H(t_1) Q_H(t_2)] | q', t' \rangle_H $$

$$ = \theta(t_2 - t_1) \int dq_1 dq_2 \langle q'', t'' | Q(t_2) | q_2, t_2 \rangle \langle q_2, t_2 | Q(t_1) | q_1, t_1 \rangle \langle q_1, t_1 | q', t' \rangle $$

$$ + \theta(t_1 - t_2) \int dq_1 dq_2 \langle q'', t'' | Q(t_1) | q_1, t_1 \rangle \langle q_1, t_1 | Q(t_2) | q_2, t_2 \rangle \langle q_2, t_2 | q', t' \rangle $$

$$ = \theta(t_2 - t_1) \int dq_1 dq_2 \, q_2 q_1 \langle q'', t'' | q_2, t_2 \rangle \langle q_2, t_2 | q_1, t_1 \rangle \langle q_1, t_1 | q', t' \rangle $$

$$ + \theta(t_1 - t_2) \int dq_1 dq_2 \, q_1 q_2 \langle q'', t'' | q_1, t_1 \rangle \langle q_1, t_1 | q_2, t_2 \rangle \langle q_2, t_2 | q', t' \rangle $$

where all operators and states are in the Heisenberg picture.

By inserting complete sets of coordinate states

$$ {}_H\langle q'', t'' | T[Q_H(t_1) Q_H(t_2)] | q', t' \rangle_H = \int_{q'}^{q''} [dq] \, q_1 q_2 \exp\left[\frac{i}{\hbar} \int_{t'}^{t''} dt \, L(t) \right] $$

number. not an operator

the order of the classical quantities

q_i does not matter.

"路径积分自动给出 time-ordered Green function."

Generating functionals.

Consider the modified Hamiltonian

$$H_j(P,Q) = H(P,Q) + j(t)\, O(P,Q)$$

any function of the P, Q operators

an arbitrary c-number function of time.

Then the Feynman kernel U_j

$$j(t'+l\delta t)\,\langle p_l|\,O(P,Q)\,|q_l\rangle\,(i\delta t/\hbar) = j(t'+\delta t\, l)\, O(p_l,q_l)\,(i\delta t/\hbar)$$

Taking a derivative with $j(t=\sigma)$: source at time $t=\sigma$.

$$\lim_{\delta t\to 0} \frac{i\hbar}{\delta t}\frac{\partial}{\partial j(\sigma)} U_j(t'',t') = {}_H\langle q'',t''|\, O(P_H(\sigma), Q_H(\sigma))\,|q',t'\rangle_H^{(j)}$$

the matrix element is computed in the presence of the source.

Then the time-ordered product.

$${}_H\langle q'',t''|\, T\prod_i O_H(P(t_i), Q(t_i))\,|q',t'\rangle_H$$

$$= \int [dp][dq] \prod_i O(p(t_i), q(t_i)) \exp\Big[\frac{i}{\hbar}\int_{t'}^{t''} dt\,(p\dot q - h - jO)\Big]_{j=0}$$

$$= \prod_i \Big[i\hbar\frac{\delta}{\delta j(t_i)}\Big] \int [dp][dq] \exp\Big[\frac{i}{\hbar}\int dt\,(p\dot q - h - jO)\Big]_{j=0}$$

Here the path integral with source term $jO(P,Q)$ serves as a generating functional for all time-ordered products of the operator $O(P,Q)$

For the configuration space.

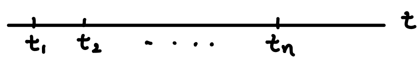
$${}_H\langle q'',t''|\, T\prod_i Q_H(t_i)\,|q',t'\rangle_H$$

$$= \prod_i \Big(i\hbar\frac{\delta}{\delta j(t_i)}\Big)\int_{q'}^{q''}[dq]\exp\Big\{\frac{i}{\hbar}\int_{t'}^{t''} dt\,\big[L(q(t),\dot q(t)) - j(t)\,q(t)\big]\Big\}_{j=0}$$

Variational partial derivative :

define functional $F[f(t)]$, with the function $f(t)$

divide the time interval into small slices



Then

$$F[f(t)] \longrightarrow F[\underbrace{f(t_1)}_{\equiv f_1}, \cdots, \underbrace{f(t_n)}_{\equiv f_n}]$$

The integral : $\int dt\, f(t) \longrightarrow \varepsilon \sum_i f_i$

The Dirac delta function : $\delta(t-t') \longrightarrow \frac{1}{\varepsilon} \delta_{ij} \Rightarrow \delta(0) = \frac{1}{\varepsilon}$

i, j correspond t and t'

Functional derivative : $\frac{\delta f(t)}{\delta f(t')} = \delta(t-t') \longrightarrow \frac{\partial f_i}{\partial f_j} = \delta_{ij}$

so $\frac{\delta}{\delta f(t)} \longrightarrow \frac{\partial}{\varepsilon \partial f_j}$

$$\frac{\delta F[f]}{\delta f(t)} = \int dt' \frac{\delta F[f]}{\delta f(t')} \times \frac{\delta f(t')}{\delta f(t)} \Rightarrow \frac{\partial F[f_1 \cdots f_n]}{\varepsilon\, \partial f_i} = \varepsilon \sum_j \frac{\partial F[f_1 \cdots f_n]}{\varepsilon \partial f_j} \frac{\partial f_j}{\varepsilon \partial f_i}$$

3.2 Functional quantization of scalar fields

the Hamiltonian :

$$H = \int d^3x \ \mathcal{H}(\phi(x), \pi(x))$$

Hermition scalar field.

generalization to fields :

$$\sum_n \longrightarrow \int d^3x$$

$|q, t\rangle \longrightarrow |\phi, t\rangle$: simultaneous eigenstate of all "coordinates"

$\phi_{\vec{x}}(t)$, i.e. $\hat{\phi}(t, \vec{x}) \, |\phi, t\rangle = \phi(\vec{x}) \, |\phi, t\rangle$

field operator — function (a number for every $\vec{x}$)

then the Feynman kernel:

$$\langle \phi_f, t_f \,|\, \phi_i, t_i \rangle = \int_{\substack{\hat{\phi}(t_i, \vec{x}) \to \phi_i(\vec{x}) \\ \hat{\phi}(t_f, \vec{x}) \to \phi_f(\vec{x})}} [D\phi(x)] \, [D\pi(x)] \, e^{i\int_{t_i}^{t_f} dt \int d^3x \, [\pi(x)\dot{\phi}(x) - \mathcal{H}(\phi(x), \pi(x))]}$$

quantum state

" involves only c - number , classical field configurations $\phi(x)$, $\pi(x)$"

the quantum transition amplitude from a field configuration $\phi_i(\vec{x})$ at time t_i to the configuration $\phi_f(\vec{x})$ at t_f is given by the sum over all classical field configurations with the weight specified by the integrand.

Path integral for Geen functions

$\langle \Omega | T(O(x_a)\, O(x_b) \cdots) | \Omega \rangle$ ← vacuum state $|\Omega ; in\rangle$ or $|\Omega : out\rangle$

↑ field operators $\phi(x_a)$ or products at the same point $\phi^n(x_a)$

$$= \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \int \prod_{\vec{x}} d\phi_i(\vec{x}) \prod_{\vec{x}} d\phi_f(\vec{x}) \quad \langle \Omega | \phi_f, t_f \rangle$$

↑ insert a complete set of coordinate eigenstates at t_i, t_f

$$\times \langle \phi_f, t_f | T(O(x_a)\, O(x_b) \cdots) | \phi_i, t_i \rangle \langle \phi_i, t_i | \Omega \rangle$$

"corresponds to an integral over all initial and final configurations"

$$= \int [D\phi(x)][D\pi(x)] \quad \langle \Omega | \phi(t_f = \infty, \vec{x}), +\infty \rangle \langle \phi(t_i = -\infty, \vec{x}), -\infty | \Omega \rangle$$

$$\times e^{i\int d^4x\, (\pi(x)\dot{\phi}(x) - \mathcal{H}(\phi(x), \pi(x)))} \quad O(x_a)\, O(x_b) \cdots$$

To obtain the final result, we compute $\langle \phi ; \pm\infty | \Omega \rangle$. Since $\phi(\vec{x})$ are the "coordinates" of the system, this is the wavefunction of the vacuum state in the coordinate representation. [Compare $\psi(\vec{x}) = \langle \vec{x} | \psi \rangle$ in quantum mechanics]

We assume that for $t \to \pm\infty$ the interacting field behaves as a free field

Then

$$\hat{\phi}(x) \xrightarrow[t \to -\infty]{} \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_p}} (e^{-ip\cdot x}\, a_p^{in} + e^{ip\cdot x}\, a_p^{+, in}) \qquad \text{with } p^0 = \omega_p$$

$$\hat{\phi}(x) \xrightarrow[t \to +\infty]{} \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_p}} (e^{-ip\cdot x}\, a_p^{out} + e^{ip\cdot x}\, a_p^{+, out})$$

and $\hat{\pi}(x) = \dot{\hat{\phi}}(x)$ (for $t \to \pm\infty$)

The above allows us to write:

$$a_p^{in\,(out)} = \lim_{t\to-\infty(+\infty)} \int d^3x \; e^{ip\cdot x} \sqrt{\frac{\omega_p}{2}} \left[\hat{\phi}(x) + \frac{i}{\omega_p} \hat{\pi}(x) \right]$$

See: $$\hat{\pi}(x) = \dot{\hat{\phi}}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} \left[a_p e^{-ip\cdot x} (-i\omega_p) + a_p^+ e^{ip\cdot x} (i\omega_p) \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(-i\sqrt{\frac{\omega_p}{2}}\right) \left[a_p e^{-ip\cdot x} - a_p^+ e^{ip\cdot x} \right]$$

$$\hat{\phi}(x) + \frac{i}{\omega_p}\hat{\pi}(x) = \int \frac{d^3q}{(2\pi)^3} \left(2 a_q e^{-iq\cdot x} \sqrt{\frac{\omega_q}{2}} \frac{1}{\omega_p} + \cdots \right)$$

term depends on a_q^+

$$\int d^3x \; e^{ip\cdot x} \left(\int \frac{d^3q}{(2\pi)^3\, 2} \, 2 a_q e^{-iq\cdot x} \sqrt{\frac{\omega_q}{\omega_p}} + \cdots \right)$$

$$= \int \frac{d^3q}{(2\pi)^3\sqrt{2\omega_q}} \left(a_q \int d^3x \; e^{-i(\vec{p}-\vec{q})\cdot\vec{x}} \sqrt{\frac{\omega_q}{\omega_p}} + \cdots \right)$$

$$= \int \frac{d^3q}{(2\pi)^3} a_q \, (2\pi)^3 \, \delta^{(3)}(\vec{p}-\vec{q})$$

$$= a_p$$

Since $a_p^{in\,(out)} |\Omega\rangle = 0$,

$$0 = \langle \phi, \mp\infty | \, a_p^{in/out} |\Omega\rangle = \lim_{t\to\mp\infty} \int d^3x \; e^{ip\cdot x} \sqrt{\frac{\omega_p}{2}} \langle \phi, \mp\infty | \left(\hat{\phi} + \frac{i}{\omega_p}\hat{\pi} \right) |\Omega\rangle$$

$$= \lim_{t\to\mp\infty} \int d^3x \; e^{ip\cdot x} \sqrt{\frac{\omega_p}{2}} \left[\phi(x) + \frac{1}{\omega_p} \frac{\delta}{\delta\phi(x)} \right] \langle \phi, \mp\infty | \Omega\rangle$$

$$\langle q, t | \, P(t) = -i \frac{\partial}{\partial q} \langle q, t |$$

QM	QFT
P	$\rightarrow \pi$
$\frac{\partial}{\partial q}$	$\rightarrow \frac{\delta}{\delta\phi}$

$\psi(q,t) = \langle q, t | \psi\rangle$ wave function

$\psi[\phi(\vec{x},t)] = \langle \phi, t | \psi\rangle$ wave functional

so $$\langle \phi, t | \, \pi(t, \vec{x}) \, | \psi\rangle = -i \frac{\delta}{\delta\phi(\vec{x},t)} \langle \phi, t | \psi\rangle$$

$$= -i \frac{\delta}{\delta\phi(\vec{x},t)} \psi[\phi(\vec{x},t)]$$

Hence $\langle \phi, \mp\infty | \Omega \rangle$ satisfies the 1st-order functional differential equation

$$\int d^3\vec{x}\ e^{-i\vec{p}\cdot\vec{x}} \left[\frac{\delta}{\delta\phi(\vec{x})} + \omega_p \phi(\vec{x}) \right] \langle \phi, \mp\infty | \Omega \rangle = 0$$

Compare $\left(\frac{d}{dx} + ax \right) f(x) = 0$, which is solved by $e^{-\frac{1}{2}ax^2}$

Hence try the solution

$$\langle \phi, \mp\infty | \Omega \rangle = N \exp\left[-\frac{1}{2} \int d^3x\, d^3y\ K(\vec{x},\vec{y})\, \phi(\vec{x})\, \phi(\vec{y}) \right]$$

at $t = \mp\infty$

Insert into the functional differential equation [Schwartz, 14.4]

$$\int d^3\vec{x}\, e^{-i\vec{p}\cdot\vec{x}} \left[-\int d^3y\ K(\vec{x},\vec{y})\, \phi(\vec{y}) + \omega_p \phi(\vec{x}) \right] N \exp[\cdots] \langle \phi, \pm\infty | \Omega \rangle = 0$$

$$\Rightarrow \quad \int d^3x\ e^{-i\vec{p}\cdot\vec{x}}\ K(\vec{x},\vec{y}) = \omega_p\, e^{-i\vec{p}\cdot\vec{y}}$$

$$\Rightarrow \quad K(\vec{x},\vec{y}) = \int \frac{d^3p}{(2\pi)^3}\ e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\ \omega_p$$

Then

$$\langle \Omega | \phi, +\infty \rangle \langle \phi, -\infty | \Omega \rangle = |N|^2\, e^{-\frac{1}{2}\int d^3x\, d^3y\ K(\vec{x},\vec{y})\, [\phi(\infty,\vec{x})\,\phi(\infty,\vec{y}) + \phi(-\infty,\vec{x})\,\phi(-\infty,\vec{y})]}$$

Consider

$$\lim_{\varepsilon\to 0^+} \varepsilon \int_{-\infty}^{+\infty} dt\ f(t)\, e^{-\varepsilon|t|} = \lim_{\varepsilon\to 0^+} \left[\varepsilon \int_0^{+\infty} dt\ f(t)\, e^{-\varepsilon t} + \varepsilon \int_{-\infty}^0 dt\ f(t)\, e^{\varepsilon t} \right]$$

$$= \lim_{\varepsilon\to 0^+} \left[-\int_0^{+\infty} dt\ f(t) \frac{d}{dt} e^{-\varepsilon t} + \int_{-\infty}^0 dt\ f(t) \frac{d}{dt} e^{\varepsilon t} \right]$$

$$= \lim_{\varepsilon\to 0^+} \left[-\int_0^{+\infty} dt\ \frac{d}{dt}[f(t)\, e^{-\varepsilon t}] + \int_0^{+\infty} dt\ f'(t)\, e^{-\varepsilon t} + \int_{-\infty}^0 dt\ \frac{d}{dt}[f(t)\, e^{\varepsilon t}] - \int_{-\infty}^0 dt\ f'(t)\, e^{\varepsilon t} \right]$$

$$= \lim_{\varepsilon\to 0^+} \left[2f(0) + \int_0^{+\infty} dt\ f'(t)\, e^{-\varepsilon t} - \int_{-\infty}^0 dt\ f'(t)\, e^{\varepsilon t} \right]$$

$$= f(+\infty) + f(-\infty) \qquad [\text{ any reasonable smooth fn. }]$$

" $f(t) = \phi(t,\vec{x})\, \phi(t,\vec{y})$

Then

$$|N|^2\, e^{-\frac{1}{2}\varepsilon\int d^3\vec{x}\, d^3\vec{y}\int_{-\infty}^{+\infty}dt\, e^{-\varepsilon|t|}\, K(\vec{x},\vec{y})\,\phi(t,\vec{x})\,\phi(t,\vec{y})}$$

$$= |N|^2 \exp\left[-\frac{\varepsilon}{2}\int d^3\vec{x}\, d^3\vec{y}\int_{-\infty}^{+\infty}dt\, e^{-\varepsilon|t|}\,\phi(t,\vec{x})\,\phi(t,\vec{y})\int\frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\,\omega_p\right]$$

$\varepsilon\omega_p = \varepsilon$, then $K(\vec{x},\vec{y}) = \delta^{(3)}(\vec{x}-\vec{y})$

$$= |N|^2\, e^{-\frac{1}{2}\varepsilon\int d^4x\ \phi^2(x)}$$

This follows from:

1° inserting the solution for $K(\vec{x},\vec{y})$

2° $\varepsilon\omega_p = \varepsilon$ Then K simplifies to $K(\vec{x},\vec{y}) = \delta^{(3)}(\vec{x}-\vec{y})$

3° $\varepsilon e^{-\varepsilon|t|} \to \varepsilon$ Since t-integral extends to $\pm\infty$ it is not obvious at this point that this replacement is correct. We will justify this latter when we shall see more explicitly the meaning of the "$i\varepsilon$-prescription".

This gives the Hamilton path integral formula for Green functions

$$\langle\Omega|\, T\{\, O(x_a)\,O(x_b)\cdots\}\,|\Omega\rangle = |N|^2\int[D\phi(x)]\,[D\pi(x)]\; e^{i\int d^4x\,[\,\pi(x)\,\dot\phi(x) - \mathcal{H}(x) + \frac{i\varepsilon}{2}\phi^2(x)]}$$

$$\times\ O(x_a)\ O(x_b)\cdots$$

- $|N|^2$ is not important, we normalize $\langle\Omega|\Omega\rangle = 1$

$$|N|^2\int[D\phi(x)]\,[D\pi(x)]\; e^{i\int d^4x\,[\,\pi(x)\,\dot\phi(x) - \mathcal{H}(x) + \frac{i\varepsilon}{2}\phi^2(x)]} = 1$$

- $\mathcal{H}(x)$ usually contains the term $\frac{1}{2}m^2\phi(x)^2$. In the following we do not write the $i\varepsilon\phi^2$ explicitly , but instead we assune $m^2 \to m^2 - i\varepsilon$

Lagrange version of the path integral

The integral ove $[D\pi]$ can be done if:

1° $\mathcal{H}$ is at most quadratic in the $\pi(x)$

2° $O(x)$ do not depend on $\pi(x)$

<u>The Gaussian path intergl</u>: [Proof see P420, Weinberg]

Finite-dimensional Gaussian integral. Let $Q(z) = \frac{1}{2} \underset{\substack{\uparrow \\ \text{matrix } A \text{ is symmetric \& non-singular}}}{A_{kl}} z_k z_l + B_k z_k + C$

be a quadratic form, $k, l = 1, \cdots, N$. Then

$$\int_{-\infty}^{+\infty} dz_1 \cdots dz_N \, e^{-Q(z)} = \left(\det \frac{A}{2\pi} \right)^{-1/2} e^{+\frac{1}{2} [A^{-1}]_{mn} B_m B_n - C}$$

$$= \underbrace{\left(\det \frac{A}{2\pi} \right)^{-1/2}}_{= \prod_{k=1}^{N} \sqrt{\frac{2\pi}{\lambda_k}}} e^{-Q(\bar{z})}$$

where $\bar{z} = -A^{-1}B$ is the stationary point $\left.\frac{\partial Q}{\partial z_k}\right|_{z=\bar{z}} = 0$ of the quadratic form

where λ_k are the eigenvalues of the symmetric matrix A

(a degenerate eigenvalue is counted several times)

We must assume $\det A \neq 0$

This generalizes to the Gaussian path integral, when the index k becomes continuous, i.e. $z_k \to z(t)$ and $\int \prod_{k=1}^{N} dz_k \to \int [Dz(t)]$ as well as

$A_{kl} \to A(t,t')$, $B_k \to B(t)$

$$\int [Dz(t)] \, e^{+\frac{1}{2}\int dt\, dt' \, A(t,t') z(t) z(t') - \int dt \, B(t) z(t) - C}$$

$$= \left(\det \frac{A}{2\pi} \right)^{-1/2} e^{+\frac{1}{2}\int dt\, dt' \, A^{-1}(t,t') B(t) B(t') - C}$$

- What is $A(t,t')^{-1}$? $\det A(t,t')$?

$$\sum_l A_{kl} z_l = \lambda z_k \quad \Rightarrow \quad \int dt' \, \underset{\substack{\uparrow \\ \text{integral kernel}}}{A(t,t')} z(t') = \lambda \underset{\substack{\uparrow \\ \text{eigenfunction}}}{z(t)}$$

$\mathrm{tr}\, A = \sum_n \lambda_n \qquad \det A = \prod_n \lambda_n$

These are usually infinite sums / products, so tr and det are defined when they converge.

$$\sum_l [A^{-1}]_{kl} A_{lk'} = \delta_{kk'} \quad \Rightarrow \quad \int dt \; A^{-1}(t_1, t)\, A(t, t_2) = \delta(t_1 - t_2)$$

Now we apply this to the path integral over $\pi(x)$:

$$\int [\mathcal{D}\pi(x)]\, e^{i\int d^4x\, [\pi(x)\dot{\phi}(x) - \mathcal{H}(x)]}$$

←--- quadratic form in $\pi(x)$ by assumption. Otherwise the following is not applicable

⌈ Write quadratic part as

$$-\frac{1}{2}\int d^4x\, d^4y \;\; i A[\phi(x)]\, \pi(x)\pi(y) \qquad \text{where } A[\phi(x)] \propto \delta^{(4)}(x-y)$$ ⌋

$$= [\det(2\pi i A[\phi(x)])]^{-1/2} \; e^{i\int d^4x\, (\bar{\pi}(x)\dot{\phi}(x) - \mathcal{H}(\phi(x), \bar{\pi}(x)))}$$

⌈ here $\bar{\pi}(x)$ is the stationary point of the quadratic form of $\pi(x)$ in the exponent.

$$\frac{\delta}{\delta\pi(x)}\left[\int d^4y\, (\pi(y)\dot{\phi}(y) - \mathcal{H}(\pi(y), \phi(y)))\right]$$

$$= \dot{\phi}(x) - \frac{\partial\mathcal{H}}{\partial\pi(x)} = 0$$

$$\Rightarrow \bar{\pi}(x) \text{ follows } \quad \dot{\phi}(x) = \left.\frac{\partial\mathcal{H}}{\partial\pi(x)}\right|_{\pi=\bar{\pi}} \qquad [\text{ Hamilton equations }]$$

$$\Rightarrow \bar{\pi}(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)}$$

$\bar{\pi}(x)$ is expressed in terms of ϕ and $\dot{\phi}$ ⌋

$$= [\det(2\pi i A[\phi(x)])]^{-1/2} \; e^{i\int d^4x\, \mathcal{L}(\phi(x), \partial_\mu\phi(x))}$$

∴ This gives the Lagrange version of the path integral (valid if $\mathcal{H}$ is quadratic in π)

$$\langle \Omega | T\{ O(x_a)\, O(x_b) \cdots \} | \Omega \rangle = |N|^2 \int [D\phi(x)] \, (\det A[\phi(x)])^{-1/2} \times e^{i\int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)} \, O(x_a)\, O(x_b) \cdots$$

must depend only on $\phi(x)$ not $\pi(x)$

Note : If $A[\phi(x)]$ does not depend on $\phi(x)$, $\det[A]$ can be pulled out of the path integral. E.g.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + \mathcal{L}_{int}$$

$$\Rightarrow \mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2 - \mathcal{L}_{int}$$

$$\Rightarrow A[\phi(x)] = \delta^{(4)}(x-y)$$

Then the final formula.

$$\boxed{\langle \Omega | T\{ O(x_a)\, O(x_b) \cdots \} | \Omega \rangle = \frac{\int [D\phi(x)] \, e^{iS[\phi(x)]} \, O(x_a)\, O(x_b) \cdots}{\int [D\phi(x)] \, e^{iS[\phi(x)]}}}$$

3.3 Perturbation expansion of Green functions

Two known methods to evaluate the path integral:

1) Discretize space-time : lattice field theory [only in Euclidean space]

2) Interactions are treated as perturbations:

$$\langle\Omega| T\{ O(x_a)\, O(x_b) \cdots \}|\Omega\rangle = |N|^2 \int [\mathcal{D}\phi(x)]\, e^{i\int d^4x\, \underset{\substack{\uparrow \\ \text{free Lagrangian}}}{\mathcal{L}_0}}$$

$$\times \sum_{N=0}^{\infty} \frac{1}{N!} \left(i\int d^4x\, \mathcal{L}_{int} \right)^N O(x_a)\, O(x_b) \cdots$$

$\boxed{\text{Green function of the free theory.}}$ [Schwartz 14.3]

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu\phi\, \partial^\mu\phi - m^2\phi^2 \right) = -\frac{1}{2}\phi\left(\Box + m^2 \right)\phi$$

n-point function: $G_0(x_1 \cdots x_n) \equiv \langle 0| T\{\phi(x_1) \cdots \phi(x_n)\}|0\rangle$

$$= \frac{1}{i}\frac{\delta}{\delta J(x_1)} \cdots \frac{1}{i}\frac{\delta}{\delta J(x_n)}\, Z_0[J]\Big|_{J=0}$$

where

$$Z_0[J] \equiv |N|^2 \int [\mathcal{D}\phi(x)]\, e^{i\int d^4x \left(\mathcal{L}_0 + \underset{\substack{\uparrow \\ \text{an external classical source } J(x)}}{J(x)}\phi(x) \right)}$$

is the generating functional of the free Green functions.

Recall

$$J(y) = \int d^4x\, \delta^{(4)}(x-y)\, J(x)$$

$$\frac{\delta J(x)}{\delta J(y)} = \delta^{(4)}(x-y)$$

$$\frac{1}{i}\frac{\delta}{\delta J(y)}\, e^{i\int d^4x\, J(x)\phi(x)} = \phi(y)\, e^{i\int d^4x\, J(x)\phi(x)}$$

The path integral that defines $Z_0[J]$ is Gaussian and can be evaluated as

$$Z_0[J] = |N|^2 \int [D\phi(x)] \, e^{i\int d^4x \, [-\frac{1}{2}\phi(x)(\Box + m^2)\phi(x) + J(x)\phi(x)]}$$

$$= |N|^2 \int [D\phi(x)] \, e^{-\frac{1}{2}\int d^4x \, d^4y \, \phi(x) D(x,y) \phi(y) - \int d^4x \, (-iJ(x))\phi(x)}$$

$$D(x,y) = i\delta^{(4)}(x-y)(\Box_y + m^2)$$

$$= \underbrace{|N|^2 \left(\det \frac{D(x,y)}{2\pi} \right)^{-1/2}}_{= Z_0[J=0] = 1 \quad , \text{ since } \langle 0|0\rangle = 1} \exp\left[\frac{1}{2}\int d^4x \, d^4y \, (-iJ(x)) D^{-1}(x,y) (-iJ(y)) \right]$$

$$= e^{-\frac{1}{2}\int d^4x \, d^4y \, J(x) \Delta_F(x-y) J(y)}$$

where $\Delta_F(x-y) \equiv D^{-1}(x,y)$

" Computation of Δ_F "

By definition of the inverse :

$$\int d^4z \; i\delta^{(4)}(x-z)[\Box_z + m^2] D^{-1}(z,y) = \int d^4z \; D(x,z) D^{-1}(z,y) = \delta^{(4)}(x-y)$$

$$\Rightarrow \quad [\Box_z + m^2] \, i \, \Delta_F(z-y) = \delta^{(4)}(z-y)$$

Solved by Fourier transformation

$$\int d^4x \, e^{ip\cdot(x-y)} [\Box_x + m^2] \, i \, \Delta_F(x-y) = 1$$

$(-ip^\mu)^2 = -p^2$ [integrate by part twice]

$$\Rightarrow \int d^4x \, e^{ip\cdot(x-y)} \Delta_F(x-y) = \frac{i}{p^2 - m^2}$$

$$\Rightarrow \quad \Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon} \qquad \text{[Feynman propagator]}$$

" Computation of the free n-point function "

n = 1, $$G_0(z) = \frac{1}{i}\frac{\delta}{\delta J(z)} Z_0[J]\Big|_{J=0}$$

$$= \frac{1}{i}\frac{\delta}{\delta J(z)}\left\{ e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x)\,\Delta_F(x-y)\,J(y)}\right\}\Big|_{J=0}$$

$$= \frac{1}{i}\left[-\frac{1}{2}\int d^4y\, \Delta_F(z-y)\,J(y) - \frac{1}{2}\int d^4x\, J(x)\,\Delta_F(x-z)\right] Z_0[J]\Big|_{J=0}$$

$$= 0$$

Comments : the n-point function vanishes for any odd n, since there is always one factor of J left and hence one gets zero for J = 0.

This also follows from the fact that $\int [D\phi]\, e^{iS[\phi]}$ is even under $\phi(x) \to -\phi(x)$, so the integrand with an odd number of $\phi(x_a)$ vanishes.

n = 2

$$\langle 0|\, T\{\phi(x)\phi(y)\}\,|0\rangle = \frac{1}{i^2}\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}\, e^{-\frac{1}{2}\int d^4z_1\, d^4z_2\, J(z_1)\,\Delta_F(z_1-z_2)\,J(z_2)}\Big|_{J=0}$$

$$= -\frac{\delta}{\delta J(x)}\left\{ \left[-\frac{1}{2}\int d^4z_2\, \Delta_F(y-z_2)\,J(z_2) - \frac{1}{2}\int d^4z_1\, J(z_1)\,\Delta_F(z_1-y)\right] \times \exp\left[-\frac{1}{2}\int d^4z_1\, d^4z_2\, J(z_1)\,\Delta_F(z_1-z_2)\,J(z_2)\right]\right\}_{J=0}$$

$$= \frac{1}{2}\Delta_F(y-x) + \frac{1}{2}\Delta_F(x-y) = \Delta_F(x-y)$$

to get a non-vanishing term for J = 0

need to differentiate the J in the pre-exponential factor.

$n=4$

$\phi_i \equiv \phi(x_i)$

$$\langle\Omega|\,T\{\phi_1\phi_2\phi_3\phi_4\}\,|\Omega\rangle = \frac{1}{i^4}\frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_4}\, e^{-\frac{1}{2}J_x \Delta_{Fxy} J_y}\Big|_{J=0}$$

$A_z B_z \equiv \int d^4z\, A(z)\, B(z)$

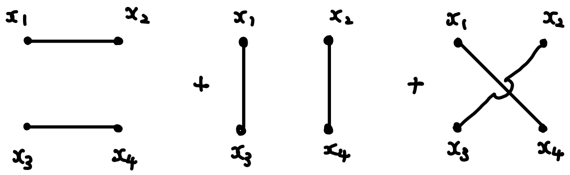
$$= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\left(-J_x\Delta_{Fx4}\, e^{-\frac{1}{2}J_x\Delta_{Fxy}J_y}\right)\Big|_{J=0}$$

$$= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\left[\left(-\Delta_{F34} + J_x\Delta_{Fx4}J_y\Delta_{Fy3}\right)e^{-\frac{1}{2}J_x\Delta_{Fxy}J_y}\right]_{J=0}$$

$$= \frac{\delta}{\delta J_1}\Big[\left(\Delta_{F34}J_z\Delta_{Fz2} + \Delta_{F24}J_y\Delta_{Fy3} + J_x\Delta_{Fx4}\Delta_{F23} - J_x\Delta_{Fx4}J_y\Delta_{Fy3}J_z\Delta_{Fz2}\right)$$
$$\times\, e^{-\frac{1}{2}J_x\Delta_{Fxy}J_y}\Big]_{J=0}$$

$$= \Delta_{F34}\,\Delta_{F12} + \Delta_{F24}\,\Delta_{F13} + \Delta_{F14}\,\Delta_{F23}$$

$$= \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 \qquad \text{where} \quad \phi_i\phi_j = \Delta_F(x_i - x_j)$$

$=$ 

Propagator of the complex scalar field

Free Lagrangian $\qquad \mathcal{L}_0 = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi$

Generating functional

$$Z_0[J, J^*] = |N|^2 \int [D\phi(x)]\,[D\phi^*(x)]\; e^{i\int d^4x\,(\mathcal{L}_0 + J\phi^* + J^*\phi)}$$

independent integration variables.

Write $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and use previous results for the real scalar field.

This gives

$$Z_0[J, J^*] = e^{-\int d^4x\, d^4y\; J(x)\,\Delta_F(x-y)\,J^*(y)}$$

no ½ here $\qquad$ Exactly the same propagator.

So generate ϕ, ϕ^* in Green function by

$$\phi(x) \longleftrightarrow \frac{1}{i}\frac{\delta}{\delta J^*(x)} \quad ; \quad \phi^*(x) \longleftrightarrow \frac{1}{i}\frac{\delta}{\delta J(x)}$$

and, in particular

$$\phi(x)\,\phi(y) = \langle 0|\, T\{\phi(x)\,\phi(y)\}\,|0\rangle = 0$$

$$\phi^*(x)\,\phi^*(y) = 0$$

$$\phi(x)\,\phi^*(y) = \langle 0|\, T\{\phi(x)\,\phi^*(y)\}\,|0\rangle = \Delta_F(x-y)$$

$$= \underset{x}{\bullet}\!\longleftarrow\!\underset{y}{\bullet}$$

"an arrow on the line to indicate that ϕ at x and ϕ^* at y"

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p\, e^{-ip\cdot x} + b_p^\dagger\, e^{ip\cdot x}\right)$$

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p^\dagger\, e^{ip\cdot x} + b_p\, e^{-ip\cdot x}\right)$$

Perturbation expansion of Green functions in the interacting theories.

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

$$Z[J] \equiv \frac{\int [D\phi]\, e^{i\int d^4x\,(\mathcal{L} + J\phi)}}{\int [D\phi]\, e^{i\int d^4x\,\mathcal{L}}}$$

$$G(x_1, \cdots, x_n) = \langle \Omega | T\{ \phi(x_1) \cdots \phi(x_n) \} | \Omega \rangle = \prod_{i=1}^{n} \frac{1}{i} \frac{\delta}{\delta J(x_i)} Z[J] \Big|_{J=0}$$

Same expansion but $\phi(x)$ is <u>not</u> a free field.

Now assume the interaction is weak. Then

$$\int [D\phi]\, e^{i\int d^4x\,\mathcal{L}}\ \phi(x_1) \cdots \phi(x_n)$$

$$= \int [D\phi]\, e^{i\int d^4x\,\underset{\substack{\uparrow \\ \text{free Lagrangian}}}{\mathcal{L}_0}} \sum_{m=0}^{\infty} \frac{1}{m!} \int d^4z_1 \cdots d^4z_m\ \underbrace{\mathcal{L}_{int}(z_1) \cdots \mathcal{L}_{int}(z_m)\, \phi(x_1) \cdots \phi(x_n)}_{\text{polynomial in the field}}$$

"Each term in this expansion is a path integral in the free theory."

From the discussion of the free theory we know that

$$\int [D\phi]\, e^{i\int d^4x\,\mathcal{L}_0} \times \text{polynomial of } \phi \quad = \text{sum of all non-vanishing contractions.}$$

– this is simply a general result for Gaussian integrals with polynomial factors

$\hookrightarrow$ $$\int [D\phi]\, e^{i\int d^4x\,\mathcal{L}}\ \phi(x_1) \cdots \phi(x_n)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \int d^4z_1 \cdots d^4z_m \times \left[\begin{array}{l} \text{sum of all contractions of fields} \\ \text{in } \mathcal{L}_{int}(z_1) \cdots \mathcal{L}_{int}(z_m)\, \phi(x_1) \cdots \phi(x_n) \end{array} \right]$$

Since $\mathcal{L}_{int}$ is small, the sum can be truncated at the order of the perturbation expansion desired.

Example : 2-point function in ϕ^3 - theory , i.e. $\mathcal{L}_{int} = -\frac{g}{3!}\phi^3$ [$g \ll 1$]

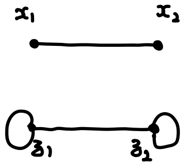
$O(g^0)$: 

$= \Delta_F(x_1 - x_2)$

$O(g^1)$: 0 $\left[\begin{array}{l}\text{cannot get a non-vanishing contraction of 5 fields} \\ \phi(x_1)\,\phi(x_2)\,\phi^3(z)\,g\end{array}\right]$

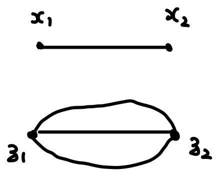
$O(g^2)$: contractions of $\phi(x_1)\,\phi(x_2)\,\phi^3(z_1)\,\phi^3(z_2)$

$\hookrightarrow$ 5 distinct possibilities

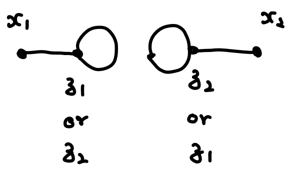
(1) 

$$\frac{1}{2}\cdot 3 \cdot 3 \cdot \left(\frac{-ig}{3!}\right)^2 \Delta_F(x_1-x_2)\int d^4z_1\, d^4z_2\, \Delta_F(z_1-z_2)\,\Delta_F(0)^2$$

from $1/m!$

(2) 

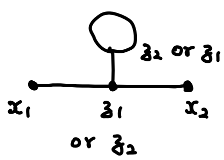
$$\frac{1}{2}\cdot 3\cdot 2\ \left(-\frac{ig}{3!}\right)^2 \Delta_F(x_1-x_2)\int d^4z_1\, d^4z_2\, \left[\Delta_F(z_1-z_2)\right]^3$$

(3) 

$$\frac{1}{2}\cdot \underset{z_1\leftrightarrow z_2}{2}\cdot 3\cdot 3\cdot \left(\frac{-ig}{3!}\right)^2 \int d^4z_1\, \Delta_F(x-z_1)\,\Delta_F(0)\int d^4z_2\, \Delta_F(x-z_2)\,\Delta_F(0)$$

$\propto G^{(1)}(x_1)\; G^{(1)}(x_2)$

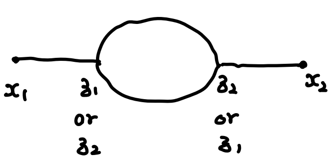
product of one-point functions at $O(g)$

(4) 

$z_1 \leftrightarrow z_2$ three possibilities to connect z_1 to z_2

$$\frac{1}{2}\cdot 2\cdot 3\cdot 3\cdot 2\quad \left(-\frac{ig}{3!}\right)^2 \int d^4z_1\, d^4z_2\, \Delta_F(x_1-z_1)\,\Delta_F(z_1-x_2)$$
$$\times\, \Delta_F(z_1-z_2)\,\Delta_F(0)$$

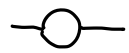
three possibilities to contract x_1 to z_1, then two for x_2 to z_1

(5) 

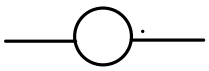
$$\frac{1}{2}\cdot 2\cdot 3\cdot 3\cdot 2\ \left(-\frac{ig}{3!}\right)^2 \int d^4z_1\, d^4z_2\, \Delta_F(x_1-z_1)\,\Delta_F(z_2-x_2)$$
$$\times\, \Delta_F(z_1-z_2)^2$$

(1), (2) are diagrams with disconnected vaccum polarization subdiagrams. They donnot contribute to the 2pt functions, because one also needs to expand the denominator in the definition of Z.

$$\int [D\phi]\, e^{i\int d^4x\, (\mathcal{L}_0 + \mathcal{L}_{int})} = 1 + $$  $+$  $+ O(g^4)$

$$\Rightarrow \underset{\text{2 pt function to } O(g^2)}{G^{(2)}(x_1, x_2)} = \underset{\substack{\| \\ \Delta_F(x_1 - x_2)}}{G^{(0)}(x_1, x_2)} + \underset{\substack{\uparrow \\ \text{disconnected part}}}{G^{(1)}(x_1)\, G^{(1)}(x_2)} + $$  $+$ 

connected part.



1PI (one particle irreducible) : cannot be disconnected by cutting through any one internel line.

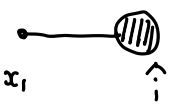
Green functions containing "composite operators"

Now $\langle \Omega | T\{ O(x_a)\, O(x_b) \cdots \} | \Omega \rangle$ where $O_a \cdots$ are not necessarily single fields, but products of fields at the same point. Such O_a are called local, composite operators.

It is evident how to generalise the rules

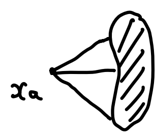
$\phi(x_1)$ $\longrightarrow$ $O_a(x_a)$

external point to which one line attaches



x_1

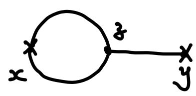
rest of the diagram.

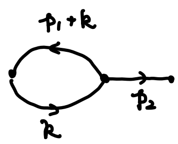


n lines if O_a contains n fields.

Again construct the sum over all contractions

Example. $-\frac{g}{3!}\phi^3$ theory

$\langle \Omega | T\{ \phi^2(x)\, \phi(y) \} | \Omega \rangle =$  at $O(g)$

and  in momentum space

[external momentum $p_1 = p_1 + k + [-k]$ at ϕ^2 vertex]

The T^* - product and comparison with the operator formalism

The derivation of the Lagrangian version of the path integral required that $O(x_a)$, $O(x_b)\cdots$ do not depend on the canonically conjugate fields. Since π contains $\dot\phi$, this means that Green functions involving $\partial_\mu\phi(x)$ are not defined so far. This is also relevant when $\mathcal{L}_{int}$ depends on $\partial_\mu\phi$, since after expansion $\partial_\mu\phi$ appears in the integrand like $O(x_a),\cdots$

Definition :

$$\langle\Omega|\, T^*\{\partial_0\phi(x)\cdots\}|\Omega\rangle \equiv \partial_0^x \langle\Omega|\, T\{\phi(x)\cdots\}|\Omega\rangle$$

only $\mu=0$ causes problems and needs definition

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon t}\Big[\langle\Omega|\,T\{\phi(x+\varepsilon t)\cdots\}|\Omega\rangle - \langle\Omega|\,T\{\phi(x)\cdots\}|\Omega\rangle\Big]$$

Note: $\langle\Omega|\,T\{\partial_0\phi(x)\cdots\}|\Omega\rangle$ exists – we only did not know how to compute it as a path integral. However, the above definition (T^*) differs from $\langle\Omega|\,T\{\partial_0\phi(x)\cdots\}|\Omega\rangle$ hence the notation T^* for a modified interpretation of time-ordering.

To see this, recall (for two fields): $\quad T\{A(x)\,B(y)\} = \theta(x^0-y^0)\,A(x)\,B(y) + \theta(y^0-x^0)\,B(y)\,A(x)$

$\hookrightarrow$ Acting with ∂_0^x outside of $T\{\cdots\}$ differs from action on A or B by the operation on the θ-function, which generates $\delta(x^0-y^0)$ terms.

Example :

$$\langle\Omega|\,T\{\partial_\mu\phi(x)\,\partial_\nu\phi(x)\}|\Omega\rangle = \underbrace{\partial_\mu^x\partial_\nu^y\langle\Omega|\,T\{\phi(x)\,\phi(y)\}|\Omega\rangle}_{=\langle\Omega|\,T^*\{\partial_\mu\phi(x)\,\partial_\nu\phi(y)\}|\Omega\rangle} - i\,\delta^0_\mu\,\delta^0_\nu\,\delta^{(4)}(x-y)$$

path integral representation

$$|N|^2\int[D\phi]\, e^{i\int d^4x\,\mathcal{L}}\,\partial_\mu\phi(x)\,\partial_\nu\phi(y)$$

$$\partial_0^x \partial_0^y \langle \Omega | T\{\phi(x)\phi(y)\} | \Omega \rangle$$

$$= \partial_0^x \partial_0^y \left[\theta(x^0-y^0)\langle \phi(x)\phi(y)\rangle + \theta(y^0-x^0)\langle \phi(y)\phi(x)\rangle \right]$$

$$= \partial_0^x \left[-\delta(x^0-y^0)\langle \phi(x)\phi(y)\rangle + \theta(x^0-y^0)\langle \phi(x)\dot\phi(y)\rangle + \delta(y^0-x^0)\langle \phi(y)\phi(x)\rangle \right.$$

$$\left. + \theta(y^0-x^0)\langle \dot\phi(y)\phi(x)\rangle \right]$$

$$= -\partial_0^x \delta(x^0-y^0)\langle \phi(x)\phi(y)\rangle - \delta(x^0-y^0)\langle \dot\phi(x)\phi(y)\rangle + \delta(x^0-y^0)\langle \phi(x)\dot\phi(y)\rangle$$

$$+ \theta(x^0-y^0)\langle \dot\phi(x)\dot\phi(y)\rangle + \partial_0^x \delta(y^0-x^0)\langle \phi(y)\phi(x)\rangle + \delta(y^0-x^0)\langle \phi(y)\dot\phi(x)\rangle$$

$$- \delta(x^0-y^0)\langle \dot\phi(y)\phi(x)\rangle + \theta(y^0-x^0)\langle \dot\phi(y)\dot\phi(x)\rangle$$

$$= -\frac{\partial}{\partial x^0}\delta(x^0-y^0)\left[\langle \phi(x)\phi(y) - \phi(y)\phi(x)\rangle \right] + \langle T\{\dot\phi(x)\dot\phi(y)\}\rangle$$

$$- \delta(x^0-y^0)\langle [\dot\phi(x), \phi(y)]\rangle + \delta(x^0-y^0)\langle [\phi(x), \dot\phi(y)]\rangle$$

$$= -\frac{\partial}{\partial x^0}\Big\{ \underbrace{\delta(x^0-y^0)\langle [\phi(x), \phi(y)]\rangle}_{=0 \quad \text{since} \quad [\phi(t,\vec{x}), \phi(t,\vec{y})] = 0} \Big\} + \delta(x^0-y^0)\langle [\dot\phi(x), \phi(y)]\rangle - \delta(x^0-y^0)\langle [\dot\phi(x), \phi(y)]\rangle$$

$$+ \delta(x^0-y^0)\langle [\phi(x), \pi(y)]\rangle + \langle T\{\dot\phi(x)\dot\phi(y)\}\rangle$$

$$= i\,\delta(x^0-y^0)\,\delta^{(3)}(\vec{x}-\vec{y}) + \langle T\{\dot\phi(x)\dot\phi(y)\}\rangle$$

$$\Rightarrow \langle T\{\dot\phi(x)\dot\phi(y)\}\rangle = \partial_0^x \partial_0^y \langle T\{\phi(x)\phi(y)\}\rangle - i\,\delta^{(4)}(x-y)$$

- In the path integral representation one always uses the T^*-product, since T is not always defined in the Lagrangian version.
- If $O(x_a)$ or $\mathcal{L}_{int}$ does not contain $\dot\phi$ or $\partial_\mu\phi$, then T^* and T coinside. In the following we denote T^* by T !

"Comparison of the path integral and operator formalisms."

Starting point $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

go to interaction picture

$$H \to H_{0,I} + V_I$$

$$\langle \Omega | T^* \{ O(x_a) O(x_b) \cdots \} | \Omega \rangle = \frac{\int [D\phi]\, e^{i\int d^4x\, \mathcal{L}}\, O(x_a) O(x_b) \cdots}{\int [D\phi]\, e^{iS}}$$

Feynman's path integral formula

- perturbation expansion involves Gaussian integrals

$$\langle \Omega | T \{ O(x_a) O(x_b) \cdots \} | \Omega \rangle = \frac{\langle 0 | T \{ O_I(x_a) O_I(x_b) \cdots e^{-i\int dt\, V_I} \} | 0 \rangle}{\langle 0 | T \{ e^{-i\int dt\, V_I} \} | 0 \rangle}$$

Gell-Mann Low formula

- perturbation expansion uses free fields in the interaction picture.

The path-integral representation is manifestly covariant because

a) $\mathcal{L}_{int}$ is, while H_{int} is not

b) T^* is, while T is not

Nevertheless both yield the same result as long as $O(x_a) \cdots$ do not depend on π's.

An intresecting situation arises, if $\mathcal{L}_{int}$ contains derivatives (but the O does not)

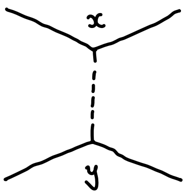
Consider as an example a scalar field with interaction

$$\mathcal{L}_{int} = -g\, J^\mu \partial_\mu \phi$$

where J^μ depends on the other fields. (if $J^\mu = \bar{\psi}\gamma^\mu\psi$, this would describe the interaction of an electron with a scalar field. Such interactions do not exist in the Standard Model, but they do in extension, such as the SUSY.)

Wanted : $\langle \Omega | T\{ J^\mu(x) J^\nu(y) \} | \Omega \rangle$ at $O(g^2)$

A) Path - integral computation



$$\int [D\phi]\, e^{i\int d^4x \mathcal{L}_0}\, g^2\, \partial_\mu \phi(x)\, \partial_\nu \phi(y)$$

which is exactly $\langle \Omega | T^* \{ \partial_\mu \phi(x)\, \partial_\nu \phi(y) \} | \Omega \rangle$

and is covariant.

B) Computation in the operator formalism

$$V_I = g J^\mu \partial_\mu \phi_I + \frac{1}{2} g^2 J_0^2$$

non-covariant term , because

$\pi = \dot{\phi} - g J^0$, but $\pi_I = \dot{\phi}_I$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} - g J^0$$

then $\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$

$$= \pi(\pi + gJ^0) - \mathcal{L}_0 + gJ^0(\pi + gJ^0) + g\vec{J}\cdot\vec{\nabla}\phi$$

$$= \pi^2 - \frac{1}{2}(\pi + gJ^0)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + 2g\pi J_0 + g^2 J_0^2 - g\vec{J}\cdot\vec{\nabla}\phi$$

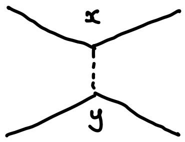
$$= \frac{1}{2}\pi^2 - \pi g J^0 - \frac{1}{2}g^2 J_0^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + 2g\pi J_0 + g^2 J_0^2 - g\vec{J}\cdot\vec{\nabla}\phi$$

$$= \mathcal{H}_0 + \underbrace{g\pi J^0 + \frac{1}{2}g^2 J_0^2 + g\vec{J}\cdot\vec{\nabla}\phi}_{V}$$

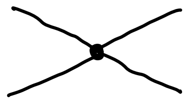
In the interaction picture, $V_I = g\dot{\phi}_I J^0 + \frac{1}{2}g^2 J_0^2 + g\vec{J}\cdot\vec{\nabla}\phi$

$$= g\,\partial_\mu \phi_I J^\mu + \frac{1}{2}g^2 J_0^2$$

Cell-Mann-Low formula gives



$+$



J_0^2

$\langle \Omega | T\{ \partial_\mu \phi(x) \partial_\nu \phi(y) \} | \Omega \rangle$

contains a non-covariant term

$-i\, \delta^0_\mu \delta^0_\nu \, \delta^{(4)}(x-y)$

local term — contributes only for $x=y$

and for $\mu=\nu=0$ due to J_0^2 :

$\Rightarrow \quad + i\, \delta^0_\mu \delta^0_\nu \, \delta^{(4)}(x-y)$

The two non-covariant terms cancel and the result is the same as in A) ∎

Consider the following correlation function:

$$\langle 0 | T \{ \exp [-i \int dt\, H_{int,I}(t)] \} | 0 \rangle$$

$$= 1 + \frac{(-ig)^2}{2} \int d^4x\, d^4y \;\; J^\mu(x) J^\nu(y) \; \langle T\{ \partial_\mu \phi(x) \partial_\nu \phi(y) \} \rangle$$

$$- i \int d^4x \, \frac{g^2}{2} \, [J^0(x)]^2$$

$$= 1 - \frac{g^2}{2} \int d^4x\, d^4y \;\; J^\mu(x) J^\nu(y) \left[\partial^x_\mu \partial^y_\nu \; \langle T\{ \phi(x) \phi(y) \} \rangle - i\, \delta^0_\mu \delta^0_\nu \, \delta^{(4)}(x-y) \right]$$

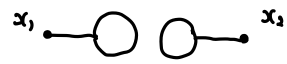
$$- i \frac{g^2}{2} \int d^4x \; d^4y \;\; J^0(x) J^0(y) \; \delta^{(4)}(x-y)$$

Connected and one-particle-irreducible (1PI) Green functions

$Z[J]$ – generating functional of n-point functions, given by all diagrams without disconnected vacuum diagrams

But $\langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\} | \Omega \rangle$ still contains disconnected contributions

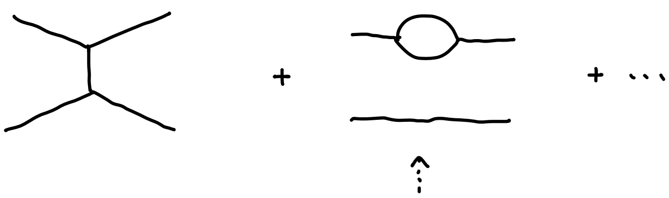
e.g.

$\langle \Omega | T\{\phi(x_1)\,\phi(x_2)\} | \Omega \rangle \supset$ 

$G^{(1)}(x_1)$ $G^{(1)}(x_2)$

Disconnected contributions are, however, "not" relavant to scattering processes.

Example in ϕ^3 theory for $2 \to 2$ scattering



does not correspond to a $2\to 2$ scattering process

Hence would like to define a generating functional directly for the connected n-point function.

The definition of the connected Green function is given by

$G(x) = G^c(x)$

$G(x_1 x_2) = G^c(x_1, x_2) + G^c(x_1)\,G^c(x_2)$ define $G^c(x_1 x_2)$

$G(x_1 x_2 x_3) = G^c(x_1 x_2 x_3) + G^c(x_1 x_2)\,G^c(x_3) + G^c(x_1 x_3)\,G^c(x_2) + G^c(x_2, x_3)\,G^c(x_1)$

$\quad + G^c(x_1)\,G^c(x_2)\,G^c(x_3)$ define $G^c(x_1 x_2 x_3)$

$\vdots$

Definition of $W[J]$: $\quad Z[J] = e^{iW[J]}$

Then $iW[J]$ is the generating functional of connected Green function, that is

$$G^c(x_1, \cdots, x_n) = \frac{1}{i}\frac{\delta}{\delta J(x_1)} \cdots \frac{1}{i}\frac{\delta}{\delta J(x_n)}\, iW[J]\Big|_{J=0}$$

See: for $n=1$

$$G(x) = \frac{1}{i}\frac{\delta}{\delta J(x)} Z[J]\Big|_{J=0} = \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\, iW[J]\right) e^{iW[J]}\Big|_{J=0} = \frac{1}{i}\frac{\delta}{\delta J(x)}\, iW[J]\Big|_{J=0}$$

$$e^{iW[0]} = Z[0] = 1$$

$$= \frac{1}{i}\frac{\delta}{\delta J(x)} \frac{\int [D\phi]\, e^{i\int d^4x\,(\mathcal{L} + J\phi)}}{\int [D\phi]\, e^{iS}}\Big|_{J=0}$$

$$= \frac{1}{i} \frac{\int [D\phi]\, e^{i\int d^4y\,(\mathcal{L} + J\phi)}\,(i\phi(x))}{\int [D\phi]\, e^{iS}}\Big|_{J=0}$$

$$= \langle \Omega | \phi(x) | \Omega \rangle$$

$$= G^c(x)$$

for $n=2$

$$G(x_1, x_2) = \frac{1}{i}\frac{\delta}{\delta J_1}\left[\left(\frac{1}{i}\frac{\delta}{\delta J_2}\, iW\right) e^{iW}\right]\Big|_{J=0}$$

$$= \left[\frac{1}{i}\frac{\delta}{\delta J_1}\frac{1}{i}\frac{\delta}{\delta J_2}\, iW + \frac{1}{i}\frac{\delta}{\delta J_1}\, iW\, \frac{\delta}{\delta J_2}\, iW\right] e^{iW}\Big|_{J=0}$$

$$= \frac{1}{i}\frac{\delta}{\delta J_1}\frac{1}{i}\frac{\delta}{\delta J_2}\, iW\Big|_{J=0} + G^c(x_1)\, G^c(x_2)$$

$$= \langle \phi(x_1)\phi(x_2)\rangle + \langle \phi(x_1)\rangle\langle \phi(x_2)\rangle$$

for $n=3$

$\vdots$

For the free scalar field:

$$iW_0[J] = \ln Z_0[J] = -\frac{1}{2}\int d^4x\, d^4y\; J(x)\, \Delta_F(x-y)\, J(y)$$

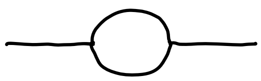
$\Rightarrow$ The only non-vanishing connected n-point function in the free theory is the 2pt function $\quad G^c(x_1\, x_2) = \Delta_F(x_1 - x_2)$

Effective action

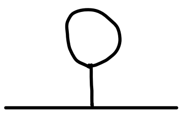
While the connected Green functions are important for scattering theory, the Green functions defined by one-particle-irreducible (1PI) diagrams are important for renormalization theory.

Definition: a diagram is 1PI if it does not fall apart into two disconnected diagrams by cutting an internal line.

Example:



1PI



not 1PI,

This is then the only contribution to the 1PI 2pt function at $O(g^2)$ in $g\phi^3$ theory.

The generating functional for the 1PI Green functions plays an important role for spontaneous symmetry breaking.

define $\phi_J(x)$ as the vacuum expectation value of the operator $\hat{\phi}(x)$ in the presence of the current J:

$$\phi_J(x) \equiv \frac{\langle \Omega | \hat{\phi}(x) | \Omega \rangle_J}{\langle \Omega | \Omega \rangle_J} = \frac{\delta}{\delta J(x)} W[J] \qquad (*)$$

can be inverted

define $J_\phi(x)$ as the current for which $(*)$ has a prescribed value $\phi(x)$:

$$\phi_J(x) = \phi(x) \quad \text{if} \quad J(x) = J_\phi(x)$$

Define the <u>effective action</u> as the Legendre transformation of $W[J]$

$$\Gamma[\phi] \equiv W[J_\phi] - \int d^4x\, J_\phi(x)\, \phi(x) \qquad (**)$$

$\Gamma[\phi]$ sum of all connected 1PI diagrams in the presence of J_ϕ (see latter)

Then we can compute

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = \int d^4y \underbrace{\left[\frac{\delta W[J]}{\delta J(y)}\right]_{J=J_\phi}}_{=\phi(y)} \frac{\delta J_\phi(y)}{\delta \phi(x)} - \int d^4y\, \frac{\delta J_\phi(y)}{\delta \phi(x)}\, \phi(y) - J_\phi(x) = -J_\phi(x)$$

Hence value of $\phi_0(x)$ are given by the sationary point of Γ,

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = 0 \qquad \text{for} \quad J=0$$

Comments:

- e.o.m. for external field ϕ, taking quantum corrections into account.
- $\Gamma[\phi]$ was introduced by Goldstone, Salam, Weinberg '62 [perturbative definition, sum of 1PI connected diagrams]
- Non-perturbative definition $(**)$ given by Witt '63
- $\phi_0 = \langle \Omega | \hat{\phi} | \Omega \rangle = 0$ in a simple theory (e.g. QED)
- $\Gamma[\phi]$ for a free scalar field $\mathcal{L} = -\frac{1}{2}\phi[\Box + m^2]\phi$

  Then $iW[J] = -\frac{1}{2}\int d^4x\, d^4y\, J(x)\, \Delta_F(x-y)\, J(y)$

  $$\phi(x) = \left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=J_\phi} = \frac{1}{\Box^2 + m^2} J_\phi(x) \quad \Rightarrow \quad J_\phi(x) = (\Box + m^2)\phi$$

  $$\Rightarrow \Gamma[\phi] = \int d^4x \left[-\frac{1}{2}\phi(\Box + m^2)\phi\right] + \cdots$$

  the effective action is identical to the classical action for a free theory

1PI diagrams from $\Gamma[\phi]$

From $\dfrac{\delta\Gamma[\phi]}{\delta\phi(x)} = -J_\phi(x)$ we have $\left.\dfrac{\delta^2\Gamma[\phi]}{\delta\phi(x)\,\delta\phi(y)}\right|_{\phi=\phi_0} = i\, G_c^{-1}(x,y)$

connected 2-pt Green fn.

See :

$$-\frac{\delta^2\Gamma[\phi]}{\delta J_\phi(y)\,\delta\phi(x)} = \delta^{(4)}(x-y)$$

$$= -\int d^4z \;\frac{\delta\phi(z)}{\delta J_\phi(y)}\;\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\,\delta\phi(y)}$$

$$= -\int d^4z \;\frac{\delta^2 W[J_\phi]}{\delta J_\phi(y)\,\delta J_\phi(z)}\;\frac{\delta^2\Gamma[\phi]}{\delta\phi(z)\,\delta\phi(y)}$$

$$= -i\;\underbrace{\left(\frac{i\,\delta^2 W}{i\delta J_\phi\; i\delta J_\phi}\right)_{yz}}_{=\;\langle\phi(y)\,\phi(z)\rangle^J_{conn}\;\equiv\;G_c^J(y,z)}\;\left(\frac{\delta^2\Gamma}{\delta\phi\;\delta\phi}\right)_{zx} \quad \longleftarrow \text{ an abstract representation}$$

To evaluate higher order derivatives of $\Gamma[\phi]$, we use the chain rule.

$$\frac{\delta}{\delta J(z)} = \int d^4w\;\frac{\delta\phi(w)}{\delta J(z)}\;\frac{\delta}{\delta\phi(w)}$$

$$= i\int d^4w\;\frac{i\,\delta^2 W[J]}{i\delta J(z)\; i\delta J(w)}\;\frac{\delta}{\delta\phi(w)}$$

$$= i\int d^4w\;G_c^J(z,w)\;\frac{\delta}{\delta\phi(w)}$$

So

$$\frac{i\,\delta^2 W[J]}{i\delta J_x\, i\delta J_y} = i\left[\frac{\delta^2\Gamma[\phi]}{\delta\phi(x)\;\delta\phi(y)}\right]^{-1}$$

$$\Rightarrow\quad \frac{i\,\delta^3 W[J]}{i\delta J_x\; i\delta J_y\; i\delta J_z} = i\int d^4w\;G_c^J(z,w)\;\frac{\delta}{\delta\phi(w)}\left[\frac{\delta^2\Gamma[\phi]}{\delta\phi(x)\;\delta\phi(y)}\right]^{-1}$$

connected 3-pt fn.

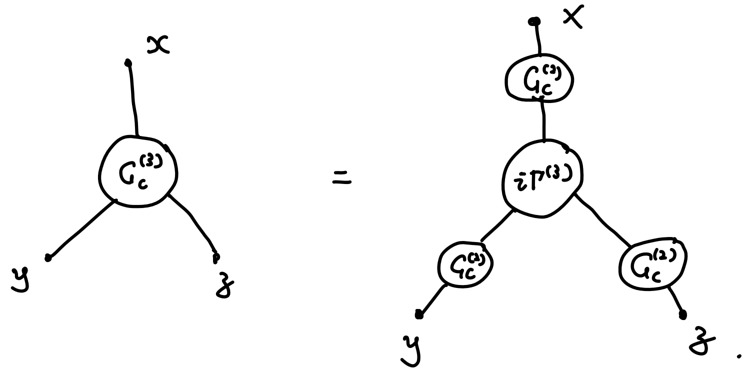
$$= i\, G^J_{c,zw} \frac{\delta}{\delta \phi_w} \left[\left(\frac{\delta^2 \Gamma}{\delta\phi\,\delta\phi} \right)^{-1} \right]_{xy}$$

$$= i\, G^J_{c,zw}\, (-1) \left[\left(\frac{\delta^2 \Gamma}{\delta\phi\,\delta\phi} \right)^{-1} \right]_{xu} \left[\frac{\delta^3 \Gamma}{\delta\phi_w\,\delta\phi\,\delta\phi} \right]_{uv} \left[\left(\frac{\delta^2 \Gamma}{\delta\phi\,\delta\phi} \right)^{-1} \right]_{vy}$$

$$= i\, G^J_{c,zw}\, (-1)\, (-i)\, G^J_{c,xu}\, (-i)\, G^J_{c,vy} \frac{\delta^3 \Gamma}{\delta\phi_w\,\delta\phi_u\,\delta\phi_v}$$

$$= i\, G^J_{c,zw}\, G^J_{c,xu}\, G^J_{c,vy} \frac{\delta^3 \Gamma}{\delta\phi_w\,\delta\phi_u\,\delta\phi_v}$$

Therefore, the above eqn is expressed diagrammatically as



Therefore, the 3rd derivative of $i\Gamma[\phi]$ is just the connected correlation function with all three full propagators removed, 1PI 3pt function

$$\left. \frac{i\, \delta^3 \Gamma[\phi]}{\delta\phi(x)\,\delta\phi(y)\,\delta\phi(z)} \right|_{\phi=\phi_0} = \langle \phi(x)\,\phi(y)\,\phi(z) \rangle_{1PI}$$

3.4 Symmetries in the path integral formalism [Peskin 9.6]

ϕ_n : bosonic fields, n can be any index [e.g. n scalar fields]

$$Z[J_n] = |N|^2 \int [D\phi_n] \exp\left(i S[\phi_n] + i\int d^4x\, J_n \phi_n \right) \qquad (*)$$

Consider the infinitesimal transformation of integration variables

$$\phi_n(x) \longrightarrow \phi_n'(x) = \phi_n(x) + \varepsilon F_n[\phi_m; \underset{\substack{\uparrow \\ \text{possible explicit dependence}}}{x}]$$

Then relabel the integration variable:

$$Z[J_n] = |N|^2 \int [D\phi_n'] \exp\left(iS[\phi_n'] + i\int d^4x\, J_n \phi_n' \right)$$

$$= |N|^2 \int [D\phi_n] \left| \det \frac{\partial \phi_{n'}'(y)}{\partial \phi_n(x)} \right| \exp\left\{ iS[\phi_n] + i\int d^4x\, J_n\phi_n \right.$$

$$\left. + \int d^4x \left[i \frac{\delta S[\phi_n]}{\delta \phi_n(x)} \underbrace{\delta\phi_n(x)}_{\varepsilon F_n} + i J_n(x)\, \delta\phi_n(x) \right] + O(\varepsilon^2) \right\}$$

The Jacobian can be rewritten using $\ln \det A = \operatorname{tr} \ln A$

$$\left| \det \frac{\partial \phi_{n'}'(y)}{\partial \phi_n(x)} \right| = \exp\left[\operatorname{tr} \ln \left[\delta_{nn'}\, \delta^4(x-y) + \varepsilon \frac{\delta F_{n'}[\phi_m; y]}{\delta \phi_n} \right] \right]$$

$\longleftarrow \delta^{(4)}(x-y) = \Omega^{-1} \delta_{x,y}$

Ω: space time volume

$$= 1 + \varepsilon \operatorname{tr} \frac{\delta F_{n'}[\phi_m(y); y]}{\delta \phi_n(x)} = 1 + \varepsilon \int d^4x \sum_n \frac{\delta F_n[\phi_m(x); x]}{\delta \phi_n(x)} + O(\varepsilon^2)$$

$\ln(1+x) \sim x$

$e^x \sim 1+x$

This is a trace in n, n' and the continuous index pair, x, y i.e. $\sum_x \to \int d^4x$

Then

$$Z[J_n] = |N|^2 \int [D\phi_n] \exp\left(iS[\phi] + i\int d^4x\, J_n\phi_n \right) \left\{ 1 + \varepsilon \int d^4x \left[\sum_n \frac{\delta F_n[\phi_m(x); x]}{\delta \phi_n} \right.\right.$$

$$\left.\left. + i\left(\frac{\delta S}{\delta \phi_n(x)} + J_n(x) \right) F_n[\phi_m(x); x] \right] + O(\varepsilon^2) \right\}$$

$= (*)$ [since the starting expression with ϕ_n' was only a relabelling of $(*)$]

Hence

$$\int [D\phi]\, e^{iS[\phi_n] + i\int d^4x\, J_n \phi_n} \times \int d^4x \left\{ \sum_n \frac{\delta F_n[\phi_m ; x]}{\delta \phi_n(x)} + i \left(\frac{\delta S}{\delta \phi_n(x)} + J_n(x) \right) F_n[\phi_m(x) ; x] \right\} = 0 \qquad (+)$$

contribution from the transformation of the path integral measure.

By functional differentiation with respect to J_n one can generate identities between Green functions.

$$\boxed{\text{Equation of motion for Green functions}}$$

Classical equation of motion

$$\frac{\delta S[\phi_m]}{\delta \phi_n} = \frac{\partial \mathcal{L}}{\partial \phi_n(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n(x))} = 0$$

e.g. for $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) - \frac{\lambda}{4!}\phi^4$: $(\partial^2 + m^2)\phi + \frac{\lambda}{3!}\phi^3 = 0$

In QFT $\frac{\delta S}{\delta \phi_n(x)}$ is an operator. The e.o.m. says that this operator vanishes in the sense that all its matrix elements in the space of states (Fock space) vanish. But what about

$$\langle \Omega | T\left\{ \frac{\delta S}{\delta \phi_n(x)} \phi_{n_1}(x_1) \phi_{n_2}(x_2) \cdots \phi_{n_N}(x_N) \right\} |\Omega\rangle = 0 \ ?$$

Answer : almost true, but not if x coincides one of the x_i

To see this, we use

$$\int [D\phi]\, e^{iS[\phi_n] + i\int d^4x\, J_n \phi_n} \times \int d^4x \left\{ \sum_n \frac{\delta F_n[\phi_m; x]}{\delta \phi_n(x)} + i\left(\frac{\delta S}{\delta \phi_n(x)} + J_n(x) \right) F_n[\phi_m(x); x] \right\} = 0$$

for $\phi'_n(x) = \phi_n(x) + \varepsilon_n(x)$. i.e. $\varepsilon F_n[\phi_m; x] = \varepsilon_n(x)$

- F_n independent of ϕ_m $\Rightarrow$ $\sum_n \frac{\delta F_n}{\delta \phi_n}$ term vanishes.
- $\varepsilon_n(x)$ arbitrary function $\Rightarrow$ The integrand of $\int d^4x$ must vanish

$$\Rightarrow \quad 0 = \int [D\phi]\, e^{iS[\phi_n] + i\int d^4x\, J_n \phi_n} \left[\frac{\delta S}{\delta \phi_n(x)} + J_n(x) \right]$$

Now take $\frac{1}{i}\frac{\delta}{\delta J_{n_1}(x_1)} \cdots \frac{1}{i}\frac{\delta}{\delta J_{n_N}(x_N)}$ then set $J_{n_i}=0$

Notation $\hat{\phi}_{n_i}(x_i)$ means that this field is omitted in the list.

Get :

$$\langle\Omega| T\left\{ \frac{\delta S}{\delta\phi_n(x)} \phi_{n_1}(x_1)\cdots\phi_{n_N}(x_N)\right\}|\Omega\rangle = \sum_{i=1}^{N} (+i)\,\delta^{(4)}(x-x_i)\,\delta_{n n_i}\,\langle\Omega|T\{\phi_{n_1}(x_1)\cdots\hat{\phi}_{n_i}(x_i)\cdots\phi_{n_N}(x_N)\}|\Omega\rangle$$

The non-vanishing right-hand side arises from $\frac{1}{i}\frac{\delta J_n(x)}{\delta J_{n_i}(x_i)} = -i\,\delta^{(4)}(x-x_i)\,\delta_{n n_i}$

E.g. $\frac{\lambda}{4!}\phi^4$ theory , $n=1$:

$$\langle\Omega| T\{ [-(\partial^2+m^2)\phi(x) - \frac{\lambda}{3!}\phi^3(x)]\,\phi(y)\}|\Omega\rangle = i\,\delta^{(4)}(x-y)$$

can be pulled in front of the integral (since T

in the path integral formalism means T^*)

This establishes an <u>exact</u> relation between $\langle\Omega|T\{\phi^3(x)\phi(y)\}|\Omega\rangle$ and $(\partial^2+m^2)\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$. To all order in perturbation theory the former equals a derivative of the two-point function.

Internal symmetries

$$\phi_n'(x) = \phi_n(x) + \varepsilon \theta^a F_n^a[\phi_m] \quad \longleftarrow \text{no explicit dependence on } x$$

Internal symmetry means $\phi_n'(x) = D_{nm}(\theta)\,\phi_m(x)$ but no transformation of the space-time point. i.e. no Lorentz-transformation or translation, only a rotation in field space. θ^a $a=1,\dots,\dim G$ parametrizes the transformations of the $\dim G$ dimensional symmetry group G.

Example: $n = 1,\dots,N$, i.e. N scalar fields and $U(N)$ symmetry

$$\phi_n'(x) = [e^{i\varepsilon\theta^a T^a}]_{nm}\,\phi_m(x)$$

$\hookrightarrow$ infinitesimal transformation $\quad \varepsilon\theta^a F_n[\phi_m] = \varepsilon\theta^a i\, T^a_{nm}\,\phi_m(x)$

For the moment assume that θ^a does not depend on x

Transformation is a symmetry means: $S[\phi_n'] = S[\phi_n]$

$$\hookrightarrow \quad 0 = \frac{\partial}{\partial\theta^a} S[\phi_n] = \frac{\partial}{\partial\theta^a} S[\phi_n + \varepsilon\theta^b F_n^b[\phi_m]] \underset{\text{chain rule}}{=} \int d^4x\, \frac{\delta S}{\delta\phi_n(x)}\,\varepsilon F_n^a[\phi_m]$$

$\Rightarrow$ The $\frac{\delta S}{\delta \phi_n}$ term in (+) vanishes.

Now distinguish two cases:

1) The path integral measure is not invariant under the symmetry transformation. i.e. $\sum_n \frac{\delta F_n}{\delta\phi_n} \neq 0$ in (+). One says that the symmetry is <u>anomalous</u>. The action S is invariant, but the generating functional $Z[J]$ is not, that is, the symmetry is not present in the quantum theory.

2) The path integral measure is invariant. Then (+) simplifies to

$$\int [D\phi_n]\, e^{iS[\phi_n] + i\int d^4x\, J_n\phi_n} \int d^4x\, J_n(x)\, F_n^a[\phi_m(x)] = 0$$

Taking $\frac{1}{i}\frac{\delta}{\delta J_{n_1}(x_1)} \cdots \frac{1}{i}\frac{\delta}{\delta J_{n_m}(x_m)}$ results in the <u>Ward – Takahashi identities:</u>

$$\sum_{k=1}^{m} \langle\Omega| T\{ \phi_{n_1}(x_1) \cdots \phi_{n_{k-1}}(x_{k-1})\, F^a_{n_k}[\phi_m(x_k)]\, \phi_{n_{k+1}}(x_{k+1}) \cdots \phi_{n_m}(x_m) \} |\Omega\rangle = 0$$

after setting $J_n = 0$

Example: $U(1)$ symmetry for a complex scalar field.

$$\delta\phi = i\theta\phi \qquad\qquad F_n[\phi] = \begin{cases} i\phi & \text{if } n \text{ refers to } \phi \\ -i\phi^\dagger & \text{if } n \text{ refers to } \phi^\dagger \end{cases}$$

$$\delta\phi^\dagger = -i\theta\phi^\dagger$$

$\hookrightarrow$ $\langle\Omega| T\{\phi(x_1)\phi^\dagger(x_2)\}|\Omega\rangle - \langle\Omega| T\{\phi(x_1)\phi^\dagger(x_2)\}|\Omega\rangle = 0$ trivial

$$\langle\Omega| T\{\phi(x_1)\phi(x_2)\phi^\dagger(x_3)\}|\Omega\rangle + \langle\Omega| T\{\phi(x_1)\phi(x_2)\phi^\dagger(x_3)\}|\Omega\rangle$$

$$- \langle\Omega| T\{\phi(x_1)\phi(x_2)\phi^\dagger(x_3)\}|\Omega\rangle = 0$$

$$\Rightarrow \langle\Omega| T\{\phi(x_1)\phi(x_2)\phi^\dagger(x_3)\}|\Omega\rangle = 0$$

In general: all n-point functions with an unequal number of ϕ and $\phi^\dagger$ fields vanish. In a scattering processes this implies that the charge carried by the ϕ particle is conserved, consistent with the conservation of the Noether current.

If the symmetry is anomalous, one derive anomalous Ward identities from (+) by keeping the $\sum_n \frac{\delta F_n}{\delta \phi_n}$ term.

A stronger result can be derived for non-anomalous symmetries for insertions of the Noether current into Green functions. For this purpose, assume $\theta^a = \theta^a(x)$ depends on x. The action will in general no longer be invariant under such a local transformation.

$$\int d^4x \, \frac{\delta S}{\delta \phi_n(x)} F_n^a[\phi_m(x)] \, \theta^a(x) = -\int d^4x \, [\partial^\mu j_\mu^a(x)] \, \theta^a(x) \qquad (*)$$

where

$$j^{\mu,a}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n(x))} F_n^a[\phi_m(x)] - K^{\mu,a}(x)$$

is the Noether current corresponding to the global symmetry transformation with $\theta^a(x)$

$$\int d^4x \, \frac{\delta S}{\delta \phi_n(x)} F_n^a[\phi_m(x)] \, \theta^a(x)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_n(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n(x))} \right] F_n^a[\phi_m(x)] \, \theta^a(x)$$

$$= \int d^4x \left[\underbrace{\frac{\partial \mathcal{L}}{\partial \phi_n(x)} F_n^a[\phi_m(x)] + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n(x))} \partial_\mu F_n^a[\phi_m(x)]} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n(x))} F_n^a[\phi_m(x)] \right] \right] \theta^a(x)$$

$$= \frac{\partial}{\varepsilon \, \partial \theta^a} \mathcal{L}(\phi_n + \varepsilon \theta^a F^a, \partial_\mu(\phi_n + \varepsilon \theta^a F^a)) \qquad \longleftarrow \theta^a : x \text{ independent}$$

$$= \frac{1}{\varepsilon} [\mathcal{L}(\phi_n', \partial_\mu \phi_n') - \mathcal{L}(\phi_n, \partial_\mu \phi_n)]$$

$$= \frac{1}{\varepsilon} \delta_a \mathcal{L} \equiv \partial_\mu K^{\mu a}$$

$$= \int d^4x \, \partial_\mu \left[K^{\mu a}(x) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n(x))} F_n^a[\phi_m(x)] \right] \theta^a(x)$$

For field configurations satisfying the field equations $\left(\frac{\delta S}{\delta \phi_n} = 0 \right)$ it immediately follows $\partial^\mu j_\mu^a = 0$, since $\theta^a(x)$ is an arbitrary function of x. Hence the above provided an alternative derivation of Noether's theorem.

Now use $(*)$ in $(+)$, and assume the global symmetry is non-anomalous. Then

$$\int [D\phi] \, e^{iS + i\int d^4x \, J_n \phi_n} \left\{ -\partial^\mu j_\mu^a[\phi_n(x)] + J_n(x) F_n^a[\phi_m(x)] \right\} = 0$$

where we used that $\theta^a(x)$ is an arbitrary fn of x, so that the integrand of $\int d^4x$ must vanish.

Taking m derivatives $\frac{1}{i}\frac{\delta}{\delta J_m(x)}$ gives the Ward identity

$$\partial_x^\mu \langle \Omega | T\{ j_\mu^a[\phi_n(x)]\, \phi_{n_1}(x_1) \cdots \phi_{n_m}(x_m) \} | \Omega \rangle$$

$$= (-i) \sum_{k=1}^{m} \delta^{(4)}(x - x_k) \langle \Omega | T\{ \phi_{n_1}(x_1) \cdots \phi_{n_{k-1}}(x_{k-1})\, F_{n_k}^a[\phi_n(x_k)]\, \phi_{n_{k+1}}(x_{k+1}) \cdots \phi_{n_m}(x_m) \} | \Omega \rangle$$

i.e. like for the equation of motion $\frac{\delta S}{\delta \phi_n} = 0$, the conservation of the current $\partial^\mu j_\mu^a = 0$ holds for the insertion into the Green functions up to δ-fn terms where x coincides with one of the x_k. These δ-fn terms are called "contact terms"

Note: in the above it matters that T actually means T^* otherwise we could not pull ∂^μ in front of the T symbol.