# WKB and Semiclassical Approximation
# Quantum Mechanics II

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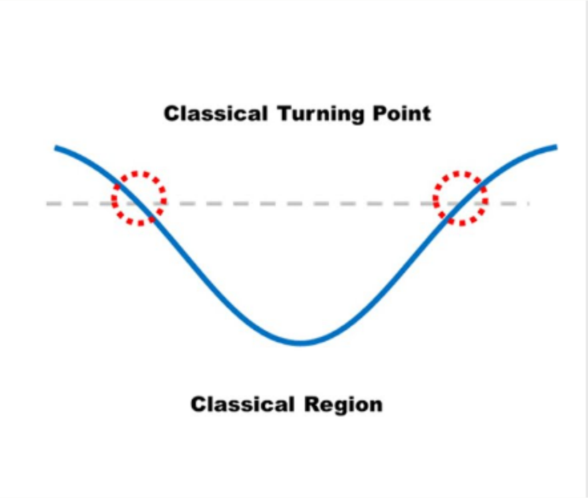

# Contents

# 1 The classical limit

Our discussion in this section will focus on one-dimensional problems. Consider a particle of mass $m$ and total energy $E$ moving in a potential $V(x)$. In classical physics, $E - V(x)$ is the kinetic energy of the particle at $x$. This kinetic energy depends on position.

## Definitions and Key Equations

- Kinetic energy: $\frac{p^2}{2m}$, define *local momentum* $p(x)$:

$$p^2(x) \equiv 2m[E - V(x)]. \tag{1}$$

- Local de Broglie wavelength $\lambda(x)$:

$$\lambda(x) \equiv \frac{h}{p(x)} = \frac{2\pi\hbar}{p(x)}. \tag{2}$$

- Time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = [E - V(x)]\psi(x). \tag{3}$$

- In terms of local momentum:

$$-\hbar^2\frac{\partial^2\psi}{\partial x^2} = p^2(x)\psi(x). \tag{4}$$

- Using momentum operator:

$$\hat{p}^2\psi(x) = p^2(x)\psi(x). \tag{5}$$

This is not an eigenvalue equation. The momentum operator acts on the wavefunction weighted by the classical, position-dependent local momentum-squared.

## Classically Allowed and Forbidden Regions

A bit of extra notation is useful. If we are in the classically allowed region, where $E > V(x)$ and $p^2(x)$ is positive, we write:

$$p^2(x) = 2m[E - V(x)] = \hbar^2 k^2(x). \tag{6}$$

introducing the local, real wavenumber $k(x)$.

If we are in the classically forbidden region, where $V(x) > E$ and $p^2(x)$ is negative, we write:

$$-p^2(x) = 2m[V(x) - E] = \hbar^2 \kappa^2(x). \tag{7}$$

introducing the local, real $\kappa(x)$.

## Wavefunctions in the WKB Approximation (3D)

The wavefunction in the WKB approximation is often written in polar form:

$$\Psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}\exp\left(\frac{i}{\hbar}\mathcal{S}(\mathbf{x},t)\right) \tag{8}$$

where $\rho(\mathbf{x},t)$ and $\mathcal{S}(\mathbf{x},t)$ are real, with $\rho$ representing the probability density:

$$\rho(\mathbf{x},t) = |\Psi(\mathbf{x},t)|^2. \tag{9}$$

Let's compute the probability current. For this we begin by taking the gradient of the wavefunction

$$\nabla\Psi = \frac{1}{2}\frac{\nabla\rho}{\sqrt{\rho}}e^{\frac{i\mathcal{S}}{\hbar}} + \frac{i}{\hbar}\nabla\mathcal{S}\Psi \tag{10}$$

We then form:

$$\Psi^*\nabla\Psi = \frac{1}{2}\nabla\rho + \frac{i}{\hbar}\rho\nabla\mathcal{S} \tag{11}$$

The current is given by

$$\mathbf{J} = \frac{\hbar}{m} \operatorname{Im}\left(\Psi^* \nabla \Psi\right). \tag{12}$$

It follows that

$$\mathbf{J} = \rho \frac{\nabla \mathcal{S}}{m}. \tag{13}$$

In classical physics a fluid with density $\rho(\mathbf{x})$ moving with velocity $\mathbf{v}(\mathbf{x})$ has a current density $\rho\mathbf{v} = \rho\frac{\mathbf{p}}{m}$. Comparing with the above expression for the quantum probability current, we deduce that

$$\mathbf{p}(\mathbf{x}) \simeq \nabla \mathcal{S} \tag{14}$$

## 2 WKB approximation scheme

To find approximate solutions for the wavefunction $\psi(x)$ in the time-independent Schrodinger equation, we represent $\psi(x)$ using a single complex function $S(x)$ as:

$$\psi(x) = \exp\left(\frac{i}{\hbar}S(x)\right), \quad S(x) \in \mathbb{C}.$$

Here, $S(x)$ has units of $\hbar$. The real part of $S$, divided by $\hbar$, gives the phase, and the imaginary part determines the magnitude of the wavefunction. Substituting into the Schrödinger equation:

$$-\hbar^2 \frac{d^2}{dx^2}\left(e^{\frac{i}{\hbar}S(x)}\right) = p^2(x) e^{\frac{i}{\hbar}S(x)},$$

and expanding the derivatives, we get:

$$-\hbar^2 \frac{d^2}{dx^2}\left(e^{\frac{i}{\hbar}S(x)}\right) = -\hbar^2\left(\frac{iS''}{\hbar} - \frac{(S')^2}{\hbar^2}\right) e^{\frac{i}{\hbar}S(x)}.$$

Simplifying and canceling the common exponential:

$$(S'(x))^2 - i\hbar S''(x) = p^2(x).$$

This non-linear equation allows us to develop an approximation scheme in powers of $\hbar$. Assuming $\hbar$ is small, we expand:

$$S(x) = S_0(x) + \hbar S_1(x) + \hbar^2 S_2(x) + \mathcal{O}(\hbar^3).$$

Substituting this into the nonlinear equation and sorting terms by powers of $\hbar$, we find:

$$\left(S_0'(x)\right)^2 - p^2(x) + \hbar \left(2S_0'S_1' - iS_0''\right) + \mathcal{O}(\hbar^2) = 0.$$

The leading order equation:

$$\left(S_0'(x)\right)^2 - p^2(x) = 0,$$

gives:

$$S_0(x) = \pm \int_{x_0}^{x} p(x')dx'.$$

The next order equation:

$$2S_0'S_1' - iS_0'' = 0,$$

leads to:

$$S_1'(x) = \frac{i}{2}\frac{p'(x)}{p(x)}.$$

Integrating:

$$S_1(x) = \frac{i}{2} \ln p(x) + C',$$

where $C'$ is a constant. Reconstructing the wavefunction:

$$\psi(x) \simeq \exp\left[\frac{i}{\hbar} S_0(x)\right] \exp\left[i S_1(x)\right] = \frac{A}{\sqrt{p(x)}} \exp\left(\pm \frac{i}{\hbar} \int_{x_0}^{x} p(x')dx'\right).$$

For the probability density:

$$\rho = \psi^* \psi = \frac{|A|^2}{p(x)} = \frac{|A|^2}{mv(x)},$$

where $v(x)$ is the local classical velocity. The probability current:

$$J(x) = \rho \frac{1}{m} \frac{\partial \mathcal{S}}{\partial x} = \frac{|A|^2}{m}.$$

## General WKB solutions

We can now construct general solutions using the basic WKB solution for both classically allowed and forbidden regions.

For the classically allowed region ($E - V(x) > 0$), where $p^2(x) = \hbar^2 k^2(x)$ and $k(x) > 0$, the solution is a superposition of waves propagating in opposite directions:

$$\psi(x) = \frac{A}{\sqrt{k(x)}} \exp\left(i \int_{x_0}^{x} k(x')\, dx'\right) + \frac{B}{\sqrt{k(x)}} \exp\left(-i \int_{x_0}^{x} k(x')\, dx'\right).$$

For the classically forbidden region ($E - V(x) < 0$), where $p^2(x) = -\hbar^2 \kappa^2(x)$ and $\kappa(x) > 0$, the solution becomes:

$$\psi(x) = \frac{C}{\sqrt{\kappa(x)}} \exp\left(\int_{x_0}^{x} \kappa(x')\, dx'\right) + \frac{D}{\sqrt{\kappa(x)}} \exp\left(-\int_{x_0}^{x} \kappa(x')\, dx'\right).$$

## Validity of the approximation

While the semiclassical approximation is derived with $\hbar$ as a small expansion parameter, it is important to understand the physical meaning of the approximation. Consider the differential equation:

$$(S_0')^2 - p^2(x) + \hbar\left(2S_0'S_1' - iS_0''\right) + \mathcal{O}\left(\hbar^2\right) = 0.$$

The $\mathcal{O}(\hbar)$ terms must be much smaller in magnitude than the $\mathcal{O}(1)$ terms. For example:

$$|\hbar S_0'S_1'| \ll |S_0'|^2.$$

Simplifying and noting $|S_0'| = |p|$:

$$|\hbar S_1'| \ll |p|,$$

and from earlier, $|S_1'| \sim |p'/p|$, so:

$$\left|\hbar\frac{p'}{p}\right| \ll |p|.$$

Rewriting this condition:

$$\left|\frac{\hbar}{p}\right|\left|\frac{dp}{dx}\right| \ll |p| \Rightarrow \lambda\left|\frac{dp}{dx}\right| \ll |p|,$$

where $\lambda = h/p$ is the de Broglie wavelength. Alternatively:

$$\left|\hbar \frac{p'}{p^2}\right| \ll 1 \Rightarrow \left|\hbar \frac{d}{dx}\frac{1}{p}\right| \ll 1,$$

or equivalently:

$$\left|\frac{d\lambda}{dx}\right| \ll 1.$$

This implies the de Broglie wavelength must vary slowly. Multiplying by $\lambda$:

$$\left|\lambda \frac{d\lambda}{dx}\right| \ll \lambda,$$

so the variation of $\lambda$ over a distance $\lambda$ must be much smaller than $\lambda$.

It is not hard to figure out what the above constraints tell us about the rate of change of the potential.

To relate this to the potential, differentiate $p^2(x) = 2m(E - V(x))$:

$$|pp'| = m\left|\frac{dV}{dx}\right| \Rightarrow \left|\frac{dV}{dx}\right| = \frac{1}{m}\left|pp'\right|.$$

Multiplying by $\lambda = h/p$, we find:

$$\left|\lambda(x)\frac{dV}{dx}\right| = \frac{2\pi\hbar}{m}\left|p'\right| \ll \frac{p^2}{m},$$

and hence:

$$\left|\lambda(x)\frac{dV}{dx}\right| \ll \frac{p^2(x)}{2m}.$$

This shows that the change in the potential over a distance equal to the de Broglie wavelength must be much smaller than the kinetic energy. This is the precise meaning of a slowly changing potential in the WKB approximation.

The slow variation conditions fail near **turning points**. At turning points, the local momentum becomes zero, and the de Broglie wavelength becomes infinite. Near a turning point $x = a$, where $V(x) - E \simeq g(x-a)$, we approximate:

$$V(x) - E \simeq g(x-a), \quad g > 0, \quad x \sim a.$$

The local momentum in the allowed region $x < a$ is:

$$p^2(x) = 2m(E - V(x)) \simeq 2mg(a-x).$$

The de Broglie wavelength is then:

$$\lambda(x) = \frac{2\pi\hbar}{p} \simeq \frac{2\pi\hbar}{\sqrt{2mg}\sqrt{a-x}}.$$

Taking the derivative:

$$\left|\frac{d\lambda}{dx}\right| \simeq \frac{\pi\hbar}{\sqrt{2mg}}\frac{1}{(a-x)^{3/2}}.$$

As $x \to a$, the right-hand side diverges, violating the condition $\left|\frac{d\lambda}{dx}\right| \ll 1$. This shows that the WKB solutions are valid only away from turning points.

Near a turning point, such as $x = a$, a "connection formula" is needed to relate WKB solutions far to the left and far to the right of the turning point.

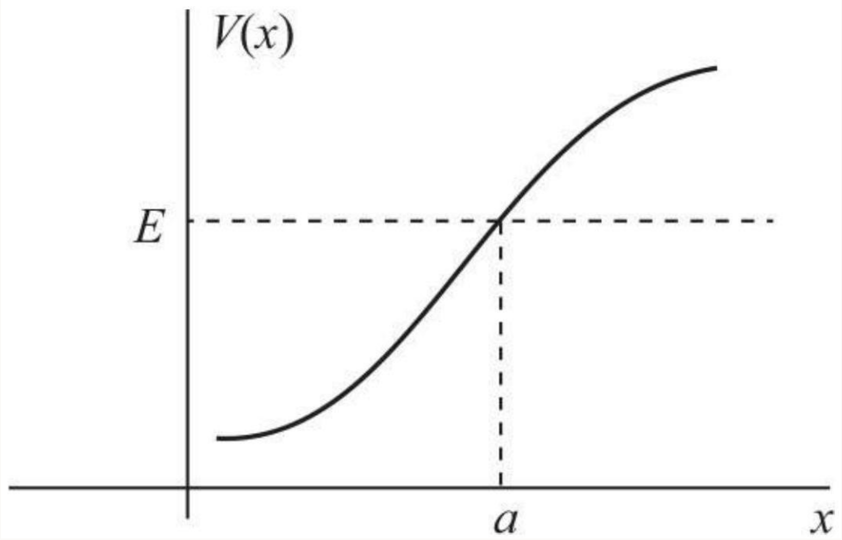


**Figure 1:** A potential $V(x)$ with a turning point at $x = a$ for a state with energy $E$.

# Example: Linear Potential and the Airy Equation

We studied the linear potential in section 6.7, solving the time-independent Schrödinger equation in momentum space using the Fourier transform. The solutions were expressed in terms of the Airy function. Here, we revisit the same linear potential:

$$V(x) = gx, \quad g > 0.$$

This analysis will be in position space. Since the potential is unbounded below, energy eigenstates are not normalizable. In section 6.7, normalization was achieved by introducing a hard wall at $x = 0$. Assume the energy $E$ is such that the classical turning point occurs at $x = a$:

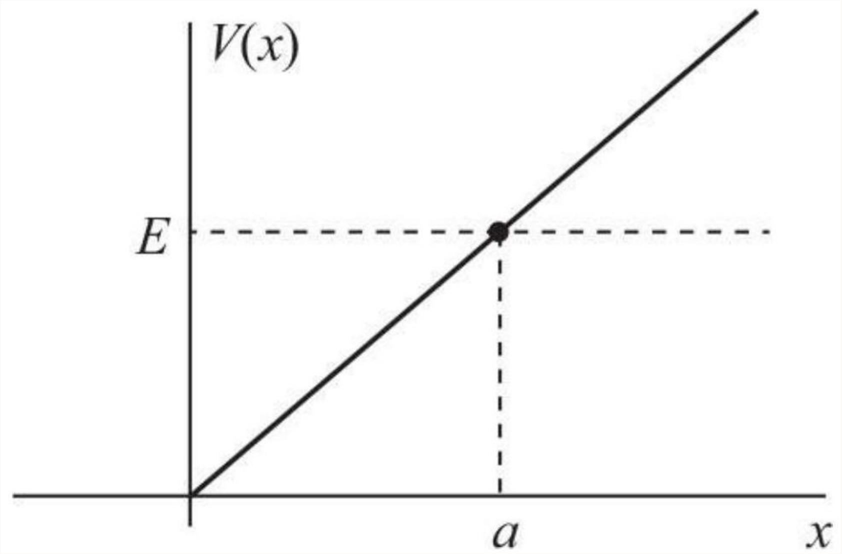


**Figure 2:** The linear potential $V(x) = gx$ and an energy eigenstate with energy $E$, where the turning point occurs at $x = a$.

The turning point is defined as:

$$V(x) - E = g(x - a), \quad a = \frac{E}{g}.$$

The classically allowed region is $x < a$, while the classically forbidden region is $x > a$. The Schrödinger equation becomes:

$$-\frac{\hbar^2}{2m}\psi'' + g(x - a)\psi = 0.$$

To remove units from this equation, let $x = L\tilde{u}$, where $\tilde{u}$ is unit-free, and $L$ has units of length:

$$-\frac{\hbar^2}{2m}\frac{1}{gL^3}\frac{d^2\psi}{d\tilde{u}^2} + \left(\tilde{u} - \frac{a}{L}\right)\psi = 0.$$

Set:

$$L^3 = \frac{\hbar^2}{2mg}, \quad u = \tilde{u} - \frac{a}{L} = \frac{1}{L}(x - a).$$

With this substitution, the differential equation simplifies to the Airy equation:

$$\frac{d^2\psi}{du^2} = u\psi.$$

The relevant solution $\psi(u)$ is the Airy function $\mathrm{Ai}(u)$, which represents the energy eigenstates of the linear potential. Using the relation between $u$, $x$, $a$, and $E$, we express the solution as:

$$\psi(u) = \mathrm{Ai}(u) = \mathrm{Ai}\left(\tilde{u} - \frac{a}{L}\right) = \mathrm{Ai}\left(\frac{1}{L}\left(x - \frac{E}{g}\right)\right).$$

These relations express the solution $\mathrm{Ai}(u)$ in terms of the physical variables. In the absence of a hard wall, all energies are allowed.

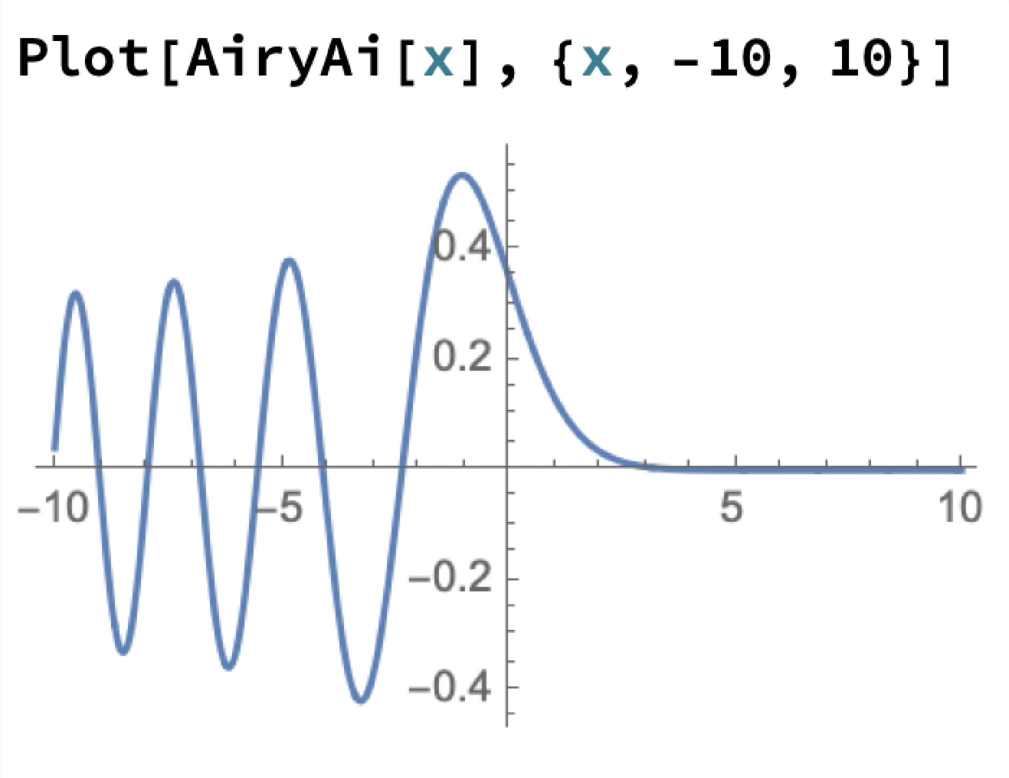


**Figure 3:** The Airy function $\mathrm{Ai}(x)$ as a function of $x$.

## Example: WKB Solutions of the Airy Equation $\psi'' = u\psi$

We reduced the problem of finding energy eigenstates for the linear potential to solving the Airy differential equation:

$$\psi''(u) = u\psi.$$

The WKB solutions for this equation are valid for $u \gg 1$ (deep in the forbidden region) and $u \ll -1$ (deep in the allowed region). These solutions are not valid near the turning point at $u = 0$.

For $u \gg 1$, set $\kappa = u^{1/2}$. Following the WKB approach, the solution is:

$$\psi(u) = \frac{C}{u^{1/4}} \exp\left[-\int_{u_0}^{u} \sqrt{u'}\, du'\right] + \frac{D}{u^{1/4}} \exp\left[\int_{u_0}^{u} \sqrt{u'}\, du'\right].$$

Here, $u_0$ is the lower limit of integration, chosen arbitrarily as long as $u_0 < u$. For convenience, we set $u_0 = 0$, giving:

$$\psi(u) = \frac{C}{u^{1/4}} \exp\left[-\int_{0}^{u} \sqrt{u'}\, du'\right] + \frac{D}{u^{1/4}} \exp\left[\int_{0}^{u} \sqrt{u'}\, du'\right].$$

Evaluating the integrals:

$$\psi(u) = \frac{C}{u^{1/4}} \exp\left[-\frac{2}{3}u^{3/2}\right] + \frac{D}{u^{1/4}} \exp\left[\frac{2}{3}u^{3/2}\right], \quad u \gg 1.$$

For $u \ll -1$, set $k = \sqrt{-u} = |u|^{1/2}$. Following the WKB approach:

$$\psi(u) = \frac{A}{|u|^{1/4}} \exp\left[i\int_u^0 \sqrt{-u'}\,du'\right] + \frac{B}{|u|^{1/4}} \exp\left[-i\int_u^0 \sqrt{-u'}\,du'\right].$$

Here, the integration limits are chosen such that the lower limit is smaller than the upper limit, inducing a minus sign in the phases. Evaluating the integrals:

$$\psi(u) = \frac{A}{|u|^{1/4}} \exp\left[i\frac{2}{3}|u|^{3/2}\right] + \frac{B}{|u|^{1/4}} \exp\left[-i\frac{2}{3}|u|^{3/2}\right], \quad u \ll -1.$$

The solutions for $u \gg 1$ and $u \ll -1$ are given by:

$$\psi(u) = \frac{C}{u^{1/4}} \exp\left[-\frac{2}{3}u^{3/2}\right] + \frac{D}{u^{1/4}} \exp\left[\frac{2}{3}u^{3/2}\right], \quad u \gg 1,$$

$$\psi(u) = \frac{A}{|u|^{1/4}} \exp\left[i\frac{2}{3}|u|^{3/2}\right] + \frac{B}{|u|^{1/4}} \exp\left[-i\frac{2}{3}|u|^{3/2}\right], \quad u \ll -1.$$

If these solutions represent a single solution of the Airy equation, there must be a relation between the coefficients $A, B, C,$ and $D$. This connection will be addressed in the section 4.

**Approximation Validity**

Our WKB solutions are approximate solutions of the Airy equation. Consider one approximate solution:

$$\psi_a(u) = \frac{1}{u^{1/4}} \exp\left(-\frac{2}{3}u^{3/2}\right).$$

Taking two derivatives, it satisfies:

$$\frac{d^2\psi_a}{du^2} = \left(u + \frac{5}{16}\frac{1}{u^2}\right)\psi_a.$$

This differs from the Airy equation by a $u^{-2}$ term, which becomes negligible for $u \gg 1$.

## 3 Using Connection Formulae

We consider the connection formulae for solutions near a turning point $x = a$, which separates a classically allowed region on the left and a classically forbidden region on the right.

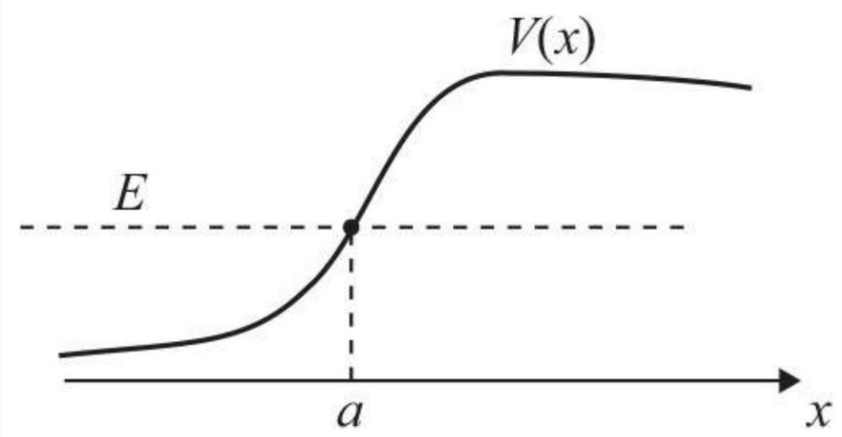


**Figure 4:** A potential $V(x)$ with a turning point at $x = a$. Connection formulae relate WKB solutions far to the left and far to the right of $x = a$.

The WKB solutions to the right are exponentials that grow or decay, while the WKB solutions to the left are oscillatory functions. These solutions connect via the following relations:

$$\frac{2}{\sqrt{k(x)}}\cos\left(\int_x^a k(x')\,dx' - \frac{\pi}{4}\right) \Longleftarrow \frac{1}{\sqrt{\kappa(x)}}\exp\left(-\int_a^x \kappa(x')\,dx'\right), \tag{15}$$

$$-\frac{1}{\sqrt{k(x)}}\sin\left(\int_x^a k(x')\,dx' - \frac{\pi}{4}\right) \Longrightarrow \frac{1}{\sqrt{\kappa(x)}}\exp\left(\int_a^x \kappa(x')\,dx'\right). \tag{16}$$

The arrows in the above relations are significant:

1. In (15), the arrow indicates that if the solution is a decaying exponential to the right of $x = a$, the corresponding solution to the left of $x = a$ is the phase-shifted cosine function that the arrow points to.

2. In (16), if the solution to the left of $x = a$ is of the oscillatory type shown, the growing exponential part to the right of $x = a$ is determined. The decaying exponential part cannot be reliably deduced.

Importantly, these connection formulae must not be applied in the direction that goes against the arrows.

## Example: Quantization Condition for a Potential with a Wall

We now attempt to find the quantization condition that governs the energies of bound states in a potential $V(x)$ that includes a hard wall at $x = 0$. Assume $V(x)$ increases monotonically and without bound, as shown in the following figure.

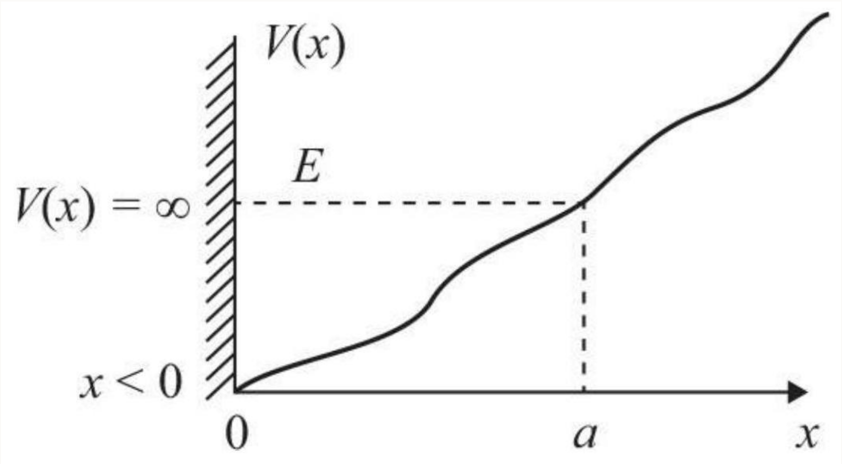


**Figure 5:** A monotonically increasing potential with a hard wall at $x = 0$. For an energy eigenstate of energy $E$, the turning point is at $x = a$.

Let $E$ denote the energy of the eigenstate. The energy and potential $V(x)$ determine the turning point $x = a$. For $x > a$, the solution must be a decaying exponential since the forbidden region extends to $x \to \infty$. The wave function for $x > a$ is of the type on the right-hand side of the connection formula (15). This accurately determines the wave function for $x \ll a$ to be:

$$\psi(x) = \frac{1}{\sqrt{k(x)}} \cos\left(\int_x^a k(x')\,dx' - \frac{\pi}{4}\right), \quad 0 \le x \ll a.$$

The wave function must vanish at the hard wall at $x = 0$. The condition $\psi(0) = 0$ requires:

$$\cos\Delta = 0, \quad \text{with} \quad \Delta \equiv \int_0^a k(x')\,dx' - \frac{\pi}{4}.$$

This is satisfied when:

$$\int_0^a k(x')\,dx' - \frac{\pi}{4} = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}.$$

The quantization condition becomes:

$$\int_0^a k(x')\,dx' = \left(n + \frac{3}{4}\right)\pi, \quad n = 0, 1, 2, \ldots \tag{17}$$

Negative integers are excluded as the left-hand side is manifestly positive. Using $k(x)$ in terms of $E$ and $V(x)$:

$$\int_0^a \sqrt{\frac{2m}{\hbar^2}\left(E - V(x')\right)}\, dx' = \left(n + \frac{3}{4}\right)\pi, \quad n = 0, 1, 2, \ldots .$$

In special cases, $a$ can be explicitly determined in terms of $E$, allowing analytical integration. More generally, numerical methods can evaluate the integral as a function of $E$, selecting energies that satisfy the quantized values.

Rewriting the wave function using $\int_x^a = \int_0^a - \int_0^x$:

$$\begin{aligned}
\psi(x) &= \frac{1}{\sqrt{k(x)}} \cos\left(\int_0^a k(x')\, dx' - \frac{\pi}{4} - \int_0^x k(x')\, dx'\right) \\
&= \frac{1}{\sqrt{k(x)}} \cos\left(\Delta - \int_0^x k(x')\, dx'\right) \\
&= \frac{1}{\sqrt{k(x)}} \sin\Delta \sin\left(\int_0^x k(x')\, dx'\right),
\end{aligned}$$

where we expanded the cosine of a sum and used $\cos\Delta = 0$. This form makes $\psi(x = 0) = 0$ explicit.

For potentials with two turning points, $a$ and $b$ ($a < b$), as shown in the following figure, the quantization condition becomes:

$$\int_a^b k(x')\,dx' = \left(n + \frac{1}{2}\right)\pi, \quad n = 0, 1, 2, \ldots.$$

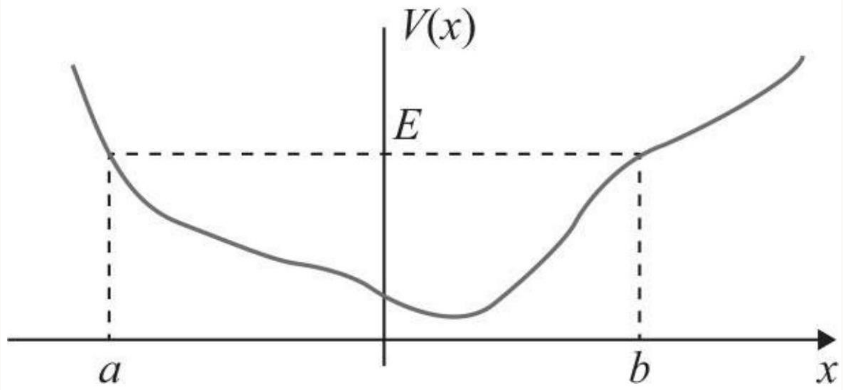


**Figure 6:** A potential for which a particle with energy $E$ encounters turning points at $x = a$ and $x = b$.

# 4 Connection formulae

Consider a general potential $V(x)$ and focus on the region around the turning point $x = a$. Near $x = a$, the potential is approximately linear, and $V(x) - E$ vanishes at $x = a$. Thus, we have:

$$V(x) - E \simeq g(x - a), \quad g > 0.$$

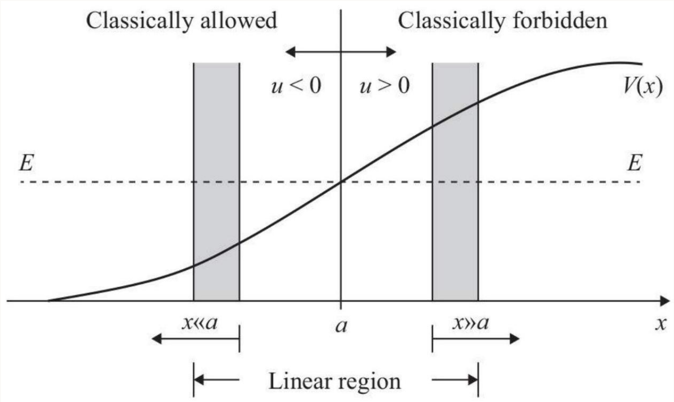


**Figure 7:** A turning point $x = a$ of a potential $V(x)$ that is approximately linear near $x = a$. The shaded regions indicate where $V(x)$ is approximately linear, and WKB expressions are valid.

The WKB basic solutions far to the right ($R$) of the turning point are:

$$\psi_R(x) = \frac{C}{\sqrt{\kappa(x)}} \exp\left(-\int_a^x \kappa(x')\,dx'\right) + \frac{D}{\sqrt{\kappa(x)}} \exp\left(\int_a^x \kappa(x')\,dx'\right), \quad x \gg a.$$

This solution was evaluated under the assumption of a strictly linear potential. In the region where $x \gg a$ and $V(x)$ is approximately linear, we use the result:

$$\psi_R(u) = \frac{C}{u^{1/4}} \exp\left(-\frac{2}{3}u^{3/2}\right) + \frac{D}{u^{1/4}} \exp\left(\frac{2}{3}u^{3/2}\right), \quad u \gg 1,$$

where $u = \frac{1}{L}(x-a)$, with $u = 0$ at the turning point.

The WKB solutions far to the left of $x = a$ are written using sines and cosines, with a convenient $\pi/4$ phase shift:

$$\psi_L(x) = \frac{A}{\sqrt{k(x)}} \cos\left(\int_x^a k(x')\,dx' - \frac{\pi}{4}\right) + \frac{B}{\sqrt{k(x)}} \sin\left(\int_x^a k(x')\,dx' - \frac{\pi}{4}\right), \quad x \ll a.$$

Using the linear potential solution, the WKB result far to the left becomes:

$$\psi_L(u) = \frac{A}{|u|^{1/4}} \cos\left[\frac{2}{3}|u|^{3/2} - \frac{\pi}{4}\right] + \frac{B}{|u|^{1/4}} \sin\left[\frac{2}{3}|u|^{3/2} - \frac{\pi}{4}\right], \quad u \ll -1.$$

From the asymptotic expansions of $\mathrm{Ai}(u)$

$$\mathrm{Ai}(u) \simeq \frac{1}{2\sqrt{\pi}}\frac{1}{u^{1/4}}\exp\left(-\frac{2}{3}u^{3/2}\right), \quad u \gg 1,$$
$$\mathrm{Ai}(u) \simeq \frac{1}{\sqrt{\pi}}\frac{1}{|u|^{1/4}}\cos\left(\frac{2}{3}|u|^{3/2} - \frac{\pi}{4}\right), \quad u \ll -1,$$

we match the solutions:

$$\frac{1}{\sqrt{\pi}}\frac{1}{|u|^{1/4}}\cos\left(\frac{2}{3}|u|^{3/2} - \frac{\pi}{4}\right) \Longleftrightarrow \frac{1}{2\sqrt{\pi}}\frac{1}{u^{1/4}}\exp\left(-\frac{2}{3}u^{3/2}\right).$$

This relation matches $C$ to $A$:

$$C = \frac{1}{2}A.$$

Similarly, from the asymptotic expansions of $\mathrm{Bi}(u)$, the matching solutions are:

$$-\frac{1}{\sqrt{\pi}}\frac{1}{|u|^{1/4}}\sin\left(\frac{2}{3}|u|^{3/2} - \frac{\pi}{4}\right) \Longleftrightarrow \frac{1}{\sqrt{\pi}}\frac{1}{u^{1/4}}\exp\left(\frac{2}{3}u^{3/2}\right).$$

This relation matches $D$ to $B$:

$$D = -B.$$

We now put it all together. Letting $A \to 2A$ and setting $C = A$, as well as $D = -B$, the WKB expressions match as follows:

The formula for $\psi_R(x)$, valid far into the forbidden region:

$$\psi_R(x) = \frac{A}{\sqrt{\kappa(x)}} \exp\left(-\int_a^x \kappa(x')\,dx'\right) - \frac{B}{\sqrt{\kappa(x)}} \exp\left(\int_a^x \kappa(x')\,dx'\right), \quad x \gg a.$$

The formula for $\psi_L(x)$, valid far into the allowed region:

$$\psi_L(x) = \frac{2A}{\sqrt{k(x)}} \cos\left(\int_x^a k(x')\,dx' - \frac{\pi}{4}\right) + \frac{B}{\sqrt{k(x)}} \sin\left(\int_x^a k(x')\,dx' - \frac{\pi}{4}\right), \quad x \ll a.$$

Up to the arrows in the connection conditions, the above equations represent the connection formulae. Note that:

- For $A = 1$ and $B = 0$, these become the relations anticipated in (15).
- For $A = 0$ and $B = -1$, these correspond to the relations in (16).